Asymptotic solutions of singular perturbed system of transport equations with small mutual diffusion in the case of many spatial variables

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Summary. We construct an asymptotic expansion on a small parameter of the solution of the Cauchy problem for a singularly perturbed system of transport equations with small nonlinearity and mutual diffusion describing the transport in a multiphase medium for many spatial variables. The asymptotic expansion of the solution is constructed as a series in powers of a small parameter and contains a functions of the boundary and inner layers. The main part of the asymptotics is described by one equation, which under certain requirements on the nonlinearity and diffusion terms is a generalization of the equation Burgers -Korteweg-de Vries in the case of many spatial variables.

Statement of the problem

The asymptotic expansion (AE) of the solution of the Cauchy problem for a singularly perturbed system of transport equations with small nonlinearity and diffusion is constructed

$$\varepsilon^2 (U_t + \sum_{i=1}^{m} D_i U_{x_i}) = AU + \varepsilon F(U) + \varepsilon^3 \sum_{i=1}^{m} B_{ij}(U) U_{x_i x_j}, |\bar{x}| < \infty, t > 0,$$

$$U(\bar{x},0) = H\omega\left(\frac{\bar{x}}{\varepsilon}\right). \qquad (1)\text{-}(2)$$

Here $U = \{u_1,\dots,u_n\}$ is the solution, $0<\varepsilon<<1$ is a small positive parameter, D_i is a diagonal constant matrix, the function $F(U)$ and the matrix $B_{ij}(U)$ are smooth enough, smooth function $\omega(x)$ is rapidly decreasing together with all derivatives. Matrix A has a single zero eigenvalue, which corresponds to the eigenvector h_0 , vector h^*_0 - eigenvector of the matrix A^T, corresponding to the zero eigenvalue, non-zero eigenvalues of the matrix A is imposed condition $Re\,\lambda <0$. Below, without limiting generality, we put $(h_0,h_0^*)=1$. Additionally, it is required that

$$\left(F(Z), h^*_0\right) = 0,\ \left(B_{ij}(Z)\right)^T h^*_0 = 0 \forall\ Z, i, j = 1,\dots,m, \operatorname{Re}\lambda > 0 \forall \lambda \neq 0. \qquad (3)$$

Such systems of equations can describe the transfer of substances in multiphase media.
The AE of the solution up to order N (determined by the smoothness of the input data) is constructed by the method of boundary functions [4] and has the form

$$U(\bar{x},t) = \sum_{i=0}^{N} \varepsilon^i (s_i(\bar{\zeta},t) + \pi_i(\bar{\xi},\tau)) + R_N = U_N + R_N,$$

$$\bar{\zeta} = (\bar{x} - \bar{V}t)/\varepsilon, \bar{\xi} = \bar{x}/\varepsilon, \tau = t/\varepsilon^2, \bar{V} = \{(D^i h_0, h_0^*)/(h_0, h_0^*), i = 1,\dots,m\}. \qquad (4)$$

The construction of AE members is described in detail in [1], [2], [3] and others. In accordance with the boundary layer method of A. V. Vasilyeva and V. F. Butuzov [4] we present nonlinear function $F(U)$ in the form

$$F(U) = F(\bar{U} + S + \Pi + R) = F(\bar{U}) + (F(\bar{U} + S) - F(\bar{U})) + (F(\bar{U} + \Pi) - F(\bar{U})) +$$

$$+ (F(\bar{U} + S + \Pi + R) - F(\bar{U} + S) - F(\bar{U} + \Pi) + F(\bar{U})) = \bar{F} + SF + \Pi F + RF.$$

A similar representation is made for $B(U)$

$$B(U) = \bar{B} + SB + \Pi B + RB.$$

Construction of asymptotic expansion of the solution

Construction regular part AE

Regular part AE have the form

$$\bar{U}(\bar{x},t) = \sum_{i=0}^{N} \varepsilon^i \bar{u}_i(\bar{x},t). \qquad (5)$$

The term $\bar{U}$ plays a supporting role.
Substitute the expantion (5) in the system (6)

$$\varepsilon^2 (\bar{U}_t + \sum_{i=1}^{m} D_i \bar{U}_{x_i}) = A\bar{U} + \varepsilon F(\bar{U}) + \varepsilon^3 \sum_{i,j=1}^{m} B_{ij}(\bar{U}) \bar{U}_{x_i x_j}, |x_i| < \infty, 1 \le i \le m,\ t > 0, \qquad (6)$$

and we obtain the equations for the terms of the expansion [4]:

$$\begin{cases} \varepsilon^0 : A\bar{u}_0 = 0, \\ \varepsilon^1 : A\bar{u}_1 = -F(\bar{u}_0), \\ \varepsilon^2 : A\bar{u}_2 = \bar{u}_{0,t} + \sum_{i=1}^{m} D_i \bar{u}_{0,x_i} - F'_u(\bar{u}_0)\bar{u}_1 \\ \quad ... \end{cases}$$

The equation at ε is solvable by condition (3).
Hence

$$\begin{cases} \bar{u}_0(x,t) = u_0(x,t)h_0 \\ \bar{u}_1(x,t) = u_1(x,t)h_0 - GF(\bar{u}_0), \end{cases}$$

Here G is a pseudo-inverse to A operator, u_0 and u_1 are some scalar functions.
We write down the condition of solvability of the equation at ε^2:

$$(\bar{u}_{0,t} + \sum_{i=1}^{m} D_i \bar{u}_{0,x_i} - F'_u(\bar{u}_0)\bar{u}_1, h*_0) = 0.$$

From condition (3) follows $(F'_u(Z)\delta, h*_0) = 0 \forall \delta.$
Therefore the solvability condition gives the equation for u_0

$$u_{0,t} + \sum_{i=1}^{m} V_i u_{0,x_i} = 0, V_i = (D_i h_0, h_0^*). \tag{7}$$

From the regular part of initial conditions $u_0(\bar{x},0) = 0$ it follows $u_0(\bar{x},t) \equiv 0 \forall \bar{x}, t.$
Similarly, all other u_i are zero. The values V_i will be used below.

Construction S function
S function have the form

$$S(\bar{\zeta},t) = \sum_{i=0}^{N} \varepsilon^i s_i(\bar{\zeta},t), \quad \zeta_i = (x_i - V_i t)/\varepsilon, i = 1,...,m. \tag{8}$$

V_i defined by the formula (7).
Function S is the solution of the system

$$\varepsilon^2(S_t + \sum_{i=1}^{m} D_i S_{x_i}) = AS + \varepsilon SF + \varepsilon^3 \sum_{i,j=1}^{m} SB_{ij} S_{x_i x_j}, \left|\bar{\zeta}\right| < \infty, t > 0. \tag{9}$$

Moving to the variables $(\bar{\zeta},t)$ taking into account $\bar{U} = 0$, we get

$$\varepsilon^2 S_t + \varepsilon \sum_{i=1}^{m} \Psi_i S_{\zeta_i} = AS + \varepsilon F(S) + \varepsilon^3 \sum_{i,j=1}^{m} B_{ij}(S) S_{\zeta_i \zeta_j}, \quad \Psi_i \equiv D_i - V_i. \tag{10}$$

Than we obtain the equations for the terms of the expansion [4]

$$\begin{cases} \varepsilon^0 : As_0 = 0, \\ \varepsilon^1 : As_1 = \sum_{i=1}^{m} \Psi_i s_{0,\zeta_i} - F(s_0) - \sum_{i,j=1}^{m} B_{ij}(s_0) s_{0,\zeta_i\zeta_j} \equiv Q_1, \\ \varepsilon^2 : As_2 = s_{0,t} + \sum_{i=1}^{m} \Psi_i s_{1,\zeta_i} - F'_u(s_0)s_1 - \sum_{i,j=1}^{m} B_{ij}(s_0) s_{1,\zeta_i\zeta_j} \equiv Q_2, \\ \quad ... \end{cases}$$

From here, taking into account the condition (3), we get

$$s_0(\bar{\zeta},t) = \varphi_0(\bar{\zeta},t)h_0,$$
$$s_1(\bar{\zeta},t) = \varphi_1(\bar{\zeta},t)h_0 + GQ_1.$$

We write down the solvability conditions of the equation at ε^2:

$$(Q_2, h*_0) = (s_{0,t} + \sum_{i=1}^{m} \Psi_i s_{1,\zeta_i} - F'_u(s_0)s_1 - \sum_{i,j=1}^{m} B_{ij} s_{1,\zeta_i\zeta_j}, h*_0) = 0.$$

Substituting here the expression for s_1 and taking into account the conditions (3) as well as equality $(\Psi_i h_0, h^*_0) = ((D_i - V_i)h_0, h^*_0) = 0,$

we obtain equation for determining φ_0. Let's introduce notation

$$M_{ij} = (\Psi_i G \Psi_j h_0, h^*_0) + (\Psi_j G \Psi_i h_0, h^*_0) \forall i \neq j, M_{ii} = (\Psi_i G \Psi_i h_0, h^*_0),$$

$$F_{i,eff}(\varphi_0) = -(\Psi_i GF(\varphi_0 h_0), h^*_0),$$

$$B_{ijk,eff}(\varphi_0) = -(\Psi_k GB_{ij}(\varphi_0 h_0)h_0, h^*_0).$$

Then the equation for determining φ_0 can be rewritten as

$$\varphi_{0,t} + \sum_{i,j=1}^{m} M_{ij}\varphi_{0,\zeta_i\zeta_j} + \sum_{i=1}^{m}(F_{i,eff}(\varphi_0))'_{\zeta_i} + \sum_{i,j,k=1}^{m}(B_{ijk,eff}(\varphi_0)\varphi_{0,\zeta_i\zeta_j})'_{\zeta_k} = 0. \tag{11}$$

In the expanded form the equation has the form

$$\varphi_{0,t} + \sum_{i,j=1}^{m} M_{ij}\varphi_{0,\zeta_i\zeta_j} + \sum_{i=1}^{m} F'_{i,eff}(\varphi_0)\varphi_{0,\zeta_i} +$$

$$+ \sum_{i,j,k=1}^{m}(B_{ijk,eff}(\varphi_0)\varphi_{0,\zeta_i\zeta_j\zeta_k} + B_{ijk,eff}{}'(\varphi_0)\varphi_{0,\zeta_i\zeta}\varphi_{0\zeta_k}) = 0. \tag{12}$$

We will impose an additional condition

$$\sum_{i,j=1}^{m} M_{ij} z_i z_j \leq 0 \forall \sum_{i=1}^{m} z_i^2 > 0.$$

To obtain the equation for s_n, $n \geq 1$, we write the expansion terms of order ε^n, ε^{n+1} and ε^{n+2} :

$$\varepsilon^j : As_j = s_{j-2,t} + Q_j, j = n, n+1, n+2,$$

where Q_1 is defined above, and Q_p for $p>1$ is expressed in terms of previously found s_{p-1}

$$Q_p = \sum_{i=1}^{m} \Psi_i s_{p-1,\zeta_i} - F'_u(s_0)s_{p-1} - \sum_{i,j=1}^{m} B_{ij} s_{p-1,\zeta_i\zeta_j}, p = 2,3,...$$

From the relations for $j=n,n+1$ it follows that

$$s_j = h_0\varphi_j(\bar{\zeta}, t) + GQ_j, j = n, n+1,$$

where φ_n, φ_{n+1} are as yet unknown functions.

Writing the solvability condition $\left(s_{n,t} + Q_{n+2}, h^*_0\right) = 0$ of the equation for s_{n+2}

after exception of s_{n+1} , we obtain the equation for s_n . Adding a designation

$$F1_{i,eff} = -(\Psi_i GF'(\varphi_0 h_0)h_0, h^*_0),$$

taking into account the notations introduced earlier, the linear equation for φ_n can be rewritten as

$$\varphi_{n,t} + \sum_{i,j=1}^{m} M_{ij}\varphi_{n,\zeta_i\zeta_j} + \sum_{i=1}^{m}(F1_{i,eff}\varphi_n)_{\zeta_i} + \sum_{i,j,k=1}^{m}(B_{ijk,eff}(\varphi_0)\varphi_{n,\zeta_i\zeta_j})_{\zeta_k} +$$

$$+ \sum_{i,j,k=1}^{m}((B_{ijk,eff}(\varphi_0))'\varphi_n\varphi_{0\zeta_i\zeta_j})_{\zeta_k} = \Phi_n(\bar{\zeta}, t), \quad n \geq 1. \tag{13}$$

where Φ_n is expressed by the previously found φ_j, $j<n$.

Construction Π function

To satisfy the initial conditions the function Π is constructed.

Π function have the form

$$\Pi(\bar{\xi}, \tau) = \sum_{i=0}^{N} \varepsilon^i p_i(\bar{\xi}, \tau), \quad \bar{\xi} = \bar{x}/\varepsilon, \tau = t/\varepsilon^2, \tag{14}$$

produced as standard [4]. Function Π is the solution of the system

$$\Pi_\tau + \varepsilon\sum_{i=1}^{m} D_i\Pi_{\xi_i} = A\Pi + \varepsilon\Pi F + \varepsilon\sum_{i,j=1}^{m} B\Pi_{ij}\Pi_{\xi_i\xi_j}, \left|\bar{\xi}\right| < \infty, \tau > 0, \tag{15}$$

together with satisfies the initial conditions and is a boundary layer function

$$S(\bar{\zeta}, 0) + \Pi(\bar{\xi}, 0) = H\omega\left(\frac{\bar{x}}{\varepsilon}\right), \Pi(\bar{\xi}, \tau) \underset{\tau\to\infty}{\to} 0.$$

The main term is defined as the solution of the system

$$p_{0,\tau} = Ap_0, \left|\bar{\xi}\right| < \infty, \tau > 0. \tag{16}$$

The initial conditions for φ_0 and p_0 are defined together, with the addition of the constraint condition p_0 at $\tau\to\infty$:

$$p_0 + \varphi_0\big|_{t=0} = U(\bar{x},0) = H\omega\left(\frac{\bar{x}}{\varepsilon}\right), \left|p_0(\bar{\xi},\infty)\right| < \infty. \tag{17}$$

From conditions (17), we obtain the initial conditions for φ_0 and p_0 .The solution of problem (16) with initial condition (17) and condition at infinity exists and satisfies the estimate

$$\left\|p_0(\bar{\xi},\tau)\right\| < C\exp(-\kappa\tau), \kappa > 0. \tag{18}$$

The remaining p_i are defined as solutions of linear inhomogeneous ODES and satisfy similar estimates:

$$p_{i,\tau} = Ap_i + P_i, i \geq 1, \left|\bar{\xi}\right| < \infty, \tau > 0.$$

$$\left\|p_i(\bar{\xi},\tau)\right\| < C\exp(-\kappa\tau), \kappa > 0. \tag{19}$$

Here P_i is expressed through previously found p_j, $j<i$.
The initial conditions for the functions φ_i and p_i are obtained together from the conditions

$$\sum_{i=1}^{\infty}\varepsilon^i(s_i(\bar{\zeta},0) + p_i(\bar{\xi},0)) = 0, \sum_{i=1}^{\infty}\varepsilon^i p_i(\bar{\xi},\tau) \underset{\tau\to\infty}{\to} 0,$$

what gives

$$s_i(\bar{\zeta},0) + p_i(\bar{\xi},0) = 0, p_i(\bar{\xi},\tau) \underset{\tau\to\infty}{\to} 0 \quad \forall i > 0. \tag{20}$$

From conditions (21) we obtain the initial conditions for the functions φ_i and p_i.

Evaluation of the residual member

The residual term is estimated by the residual term in the problem.
The question of the existence of a solution and exponential estimates of the solution of equations (1)-(2), (11)-(17) under rapidly decreasing initial conditions for the variable ζ not considered here.
The residual term was estimated by residual.
Let the function $\omega(z)$ have derivatives up to $N+3$rd order, the function $F(z)$ have derivatives up to $N+3$rd order in the domain $||U||<C$, $C>0$, and let $||U(x, 0)||<C-\delta$, $\delta>0$.
Theorem. The solution of the problem (1) - (2) is represented as

$$U(\bar{x},t) = \sum_{i=0}^{N}\varepsilon^i(s_i(\bar{\zeta},t) + p_i(\bar{\xi},\tau)) + R_N = U_N + R_N, \tag{21}$$

where $U_N = S_N + \Pi_N$ is the constructed AE, and the residual term R_N satisfies the Cauchy problem

$$\varepsilon^2(R_t + \sum_{i=1}^{m}D_iR_{x_i}) = AR + \varepsilon RF + \varepsilon^3\sum_{i,j=1}^{m}RB_{ij}R_{x_ix_j} + r, \left|\bar{x}\right| < \infty, t > 0,$$

$$R(\bar{x},0) = 0, r = O(\varepsilon^N).$$

Conclusion

1. The solution of the problem (1) - (2) at $t> t_0$, where $t_0 >0$ is some fixed (independent of ε), has the form

$$U(\bar{x},t) = \sum_{i=0}^{N}\varepsilon^i(s_i(\bar{\zeta},t) + p_i(\bar{\xi},\tau)) + R_N = s_0(\bar{\zeta},t) + O(\varepsilon) = \varphi_0(\bar{\zeta},t)h_0 + O(\varepsilon),$$

where the principal term AE $\varphi_0(\zeta, t)$ is the solution of the equation

$$\varphi_{0,t} + \sum_{i,j=1}^{m}M_{ij}\varphi_{0,\zeta_i\zeta_j} + \sum_{i=1}^{m}(F_{i,eff}(\varphi_0))'_{\zeta_i} + \sum_{i,j,k=1}^{m}(B_{ijk,eff}(\varphi_0)\varphi_{0,\zeta_i\zeta_j})'_{\zeta_k} = 0$$

(generalized Burgers – Korteweg – de Vries equation). For a quadratic function $F(u)$ and constant matrics $B(u)$, equation (13) is a generalization of the Burgers - Korteweg – de Vries equation [5] to the multidimensional case:

$$\varphi_{0,t} + \sum_{i,j=1}^{m}M_{ij}\varphi_{0,\zeta_i\zeta_j} + \sum_{i=1}^{m}k_i\varphi_0\varphi_{0,\zeta_i} + \sum_{i,j,k=1}^{m}B_{ijk,eff}\varphi_{0,\zeta_i\zeta_j\zeta_k} = 0.$$

For the case of a single spatial variable, the equation (13) differs from the BKdF equation only in the numerical values of the coefficients

$$\varphi_{0,t} + M\varphi''_{0,\zeta\zeta} + k\varphi_0\varphi'_{0,\zeta} + B\varphi'''_{0,\zeta\zeta\zeta} = 0.$$

2. Very interesting properties of the spatial part (the relationship between the degree of degeneracy of the parabolic part of the operator and the dimension of the system (1) are obtained in [1],[3].

3. For the dissipativity of the equation (11), the condition is sufficient

$$M(\bar{z}) = \sum_{i,j=1}^{m} M_{ij} z_i z_j \leq 0 \forall \sum_{i=1}^{m} z_i^2 > 0. \tag{22}$$

It is shown in [1] that the set of matrices *A* that satisfies the conditions imposed above and satisfies the condition (22) is not empty.

4. When *B (U) =0*, the system (1) becomes a system of transfer equations, i.e. a hyperbolic type system.
In this case, the equation (11) that defines the main AE term has the form

$$\varphi_{0,t} + \sum_{i,j=1}^{m} M_{ij}\varphi_{0,\zeta_i\zeta_j} + \sum_{i=1}^{m} (F_{i,eff}(\varphi_0))'_{\zeta_i} = 0, \tag{23}$$

In [1], a class of matrices *A* is allocated for which the quadratic form $M(\bar{z})$,defining the spatial part of the equation, is sign-positive, and the equation (23) i.e. becomes a parabolic equation (such as the Burgers equation). The nature of the evolution of the principal term of the AE can be described as the movement with some "effective speed" V_{eff}

$$\bar{V}_{eff} = \{V_i = (D_i h_0, h_0^*), i = 1, \dots, n\}$$

with simultaneous pseudodiffusion, the nature of which is determined by the coefficients M_{ij} , which is influenced by nonlinearity.
The spatial part with the second derivatives is determined by the symmetric matrix *M*. In [1] it is obtained that the quadratic form $M(\bar{z})$ can be degenerate, and the degree of degeneration depends on the ratio of the number of equations n and the number of spatial variables m. For a class of matrices *A* allocated in [1], for m=3 (three spatial variables) and for n=2, the matrix *M* has two zero eigenvalues and one negative, for n=3 - one zero and two negative, for n=4 and more , all eigenvalues of the matrix M become negative.
Figure 1 shows the evolution of the main AE member for m=3 and n=2, n=3.

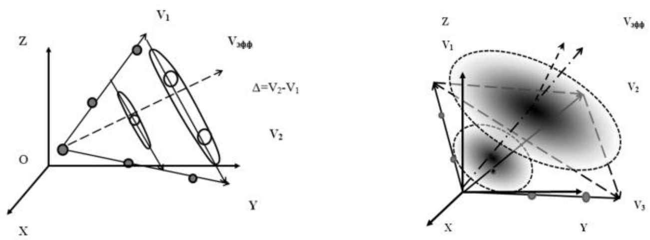

Figure 1. n=2, n=3.

In the case of three-dimensional space, the picture of the solution evolution of the principal term of the AE will have the following view. In a two-phase environment, pseudodiffusion processes develop along one axis (the direction of which is given by the vector $\Delta = \bar{V}_1 - \bar{V}_2$). Accordingly, the initial local perturbation will move in space with an "effective" average speed Veff and simultaneously deform, diffusing into a "cloud" extended in the direction of the vector Δ. In the case of three phases (a system of three equations), the initial perturbation will move to the Pro-
travel with the averaged speed and diffuse in the plane of vectors $\bar{V}_1 - \bar{V}_2, \bar{V}_1 - \bar{V}_3$, forming a flat "cloud".
In the case of four or more phases (a system of 4 or more equations), the initial perturbation will move at the effective velocity V_{ef} and diffuse over all three axes.

5. For the case of *B(U) =0* and a slightly different type of nonlinearity, an AE of a similar problem is constructed in [1]. When a number of additional conditions are imposed, it is possible to prove the estimate of the residual term of the constructed AE in the norm *C*.

6. The obtained result (11) allows us to identify non-obvious patterns of behavior of the solution of the Cauchy problem for singularly perturbed systems of type (1), as well as to identify non-obvious patterns of transfer processes in multiphase media in the case of rapid exchange between phases.
7. The numerical solution of the Cauchy problem for equation (11) requires significantly less computational resources than the solution of the original problem (1), due to the fact that equation (11) is not singularly perturbed.

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