Non-smooth Two Variable Expansions for Separation of Motions In Impact and Impulsively Loaded Oscillators

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Summary. Non-smooth two variable expansions for impact and impulsively loaded oscillators are introduced. In particular, the role of fast temporal scale is assigned to the triangle sine wave, whereas amplitude and/or frequency modulations are described in a smooth slow temporal scale. The developed tool allows for derivations of closed form solutions describing non-stationary oscillatory dynamics of systems with cyclical impacts or external impulses. Different illustrating examples of oscillators with amplitude limiters are considered assuming the coefficient of restitution is closed to unity.

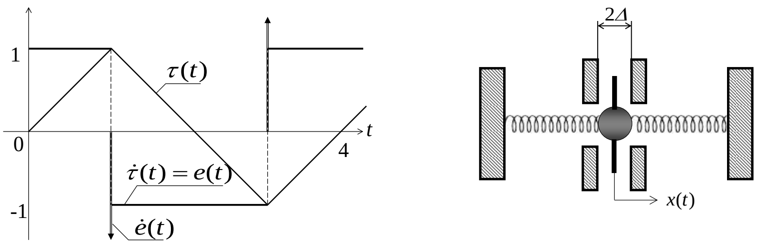
Introduction

Nonstationary dynamics of oscillating systems under non-holonomic constraint conditions became of significant interest due to different ideas of using impact oscillators as elements of energy absorbers or harvesters. In contrast to the harmonic oscillator, the basic impact oscillator has no specific natural frequency and therefore can interact with different subsystems in a wide range of spectrum, which is important property for the design of such devices. It is shown in this work that the 'hyperbolic complexification' of the state variables provides the adequate way to describing the effect of energy loss through specific boundary conditions generated by the triangle wave temporal argument. Then analytical algorithms for solving the corresponding boundary value problems are applied. In addition to the possibility of calculations without conditioning the variables at collision times, the suggested type of solutions can be effectively used in different analytical manipulations dictated by the purpose of study due to the closed form of solutions.

Technical details

The present analytical procedure essentially involves a couple of piecewise linear functions admitting also representation in a closed form through elementary functions (Fig.1a)

$$\begin{aligned} &\tau(t)=\frac{2}{\pi}\arcsin\left(\sin\left(\frac{\pi t}{2}\right)\right)=\begin{cases} t & \text{if } -1\le t\le 1 \\ 2-t & \text{if } \ 1<t\le 3 \end{cases}, \quad \tau(t+4)\overset{\forall\phi}{=}\tau(t) \\ &e(t)=\dot{\tau}(t)=\operatorname{sgn}\left(\cos\left(\frac{\pi t}{2}\right)\right)=\begin{cases} 1 & \text{if } -1<t<1 \\ -1 & \text{if } \ 1<t<3 \end{cases}, \quad e(t+4)\overset{\forall\phi}{=}e(t) \end{aligned} \tag{1}$$



(a) (b)

Figure 1: a) Non-smooth basis, and b) harmonic oscillator with amplitude limiters.

Functions (1) are used in different variations and contents in the literature, see for instance references [1, 2] as relevant to the area. However, normalizations for the amplitudes and periods shown in Fig.1 are essential for the present methodology, in which the rectangular wave $e(t)$ plays the role of a *unipotent* of the so-called *hyperbolic number* due to the property $e^2=1$. Namely any periodic process $x(t)$ of the period T = 4 admits representation in the form [3] $x(t)=X(\tau)+Y(\tau)e$. In case of modulated oscillatory motions a slow time η is added as

$$x(t)=X(\tau,\eta)+Y(\tau,\eta)e, \quad e^2=1 \tag{2}$$

where $\tau=\tau(\varphi)$, $e=e(\varphi)$, $\dot{\varphi}=\omega(\eta)$, $\eta=\varepsilon t$ $(0<\varepsilon<<1)$.

Then the triangle wave τ is considered as a fast temporal scale of the following two variable expansions

$$\begin{aligned} X(\tau,\eta)&=X_0(\tau,\eta)+X_1(\tau,\eta)\varepsilon+X_2(\tau,\eta)\varepsilon^2+O\left(\varepsilon^3\right) \\ Y(\tau,\eta)&=Y_0(\tau,\eta)+Y_1(\tau,\eta)\varepsilon+Y_2(\tau,\eta)\varepsilon^2+O\left(\varepsilon^3\right) \\ \dot{\varphi}=\omega(\eta)&=\omega_0(\eta)+\omega_1(\eta)\varepsilon+\omega_2(\eta)\varepsilon^2+O\left(\varepsilon^3\right) \end{aligned} \tag{3}$$

Substituting (2-3) into differential equation of motion and using the corresponding condition on velocity at impact times leads to the sequence of boundary value problems for X and Y on the interval of the oscillating time argument $-1\le\tau\le 1$ [4]. Some of the boundary conditions occur as a result of elimination of singularities produced by differentiation of the basic functions (1), whereas specific boundary condition is imposed by the dissipative interaction with amplitude limiters. *Note that the conventional two variable procedure, which is typically applied to quasi linear vibrating systems, produces the differential equations for slow motions as a result of elimination of secular terms from the expansions [5]. Instead the present analytical algorithm derives the slow time equations from the boundary conditions at* $\tau = \pm 1$. Due to technical complexity of analytical manipulations with series (3) in general form, different adaptations for particular cases at the preliminary stage of derivations can be applied. For instance, free impact vibrations of the model shown in Fig.1b do not have amplitude modulations as long as the energy remains sufficient for reaching the amplitude limiters. In this case, the explicit dependence on η in *X* and *Y* terms of expansions (3) can be ignored as it is done in the example below. The effect of energy loss is completely captured by the slow varying frequency terms.

Example

Free vibrations of the illustrating model, which is shown in Fig. 1b, is described with equations

$$\ddot{x} + \Omega^2 x = 0, \ |x| \le \Delta, \quad \dot{x}(t_i + 0) = -k\dot{x}(t_i - 0), \quad k = 1 - \varepsilon, \quad 0 < \varepsilon \ll 1 \tag{4}$$

This oscillator experiences impulsive reaction forces from the amplitude limiters at collision times. Therefore, assuming that the condition $|x| \le \Delta$ holds, equation (4) is replaced with

$$\ddot{x} + \Omega^2 x = pe'(\varphi), \quad p = p(\eta) = p_0(\eta) + p_1(\eta)\varepsilon + p_2(\eta)\varepsilon^2 + O\left(\varepsilon^3\right) \tag{5}$$

where the slowly varying intensity of impulses $p(\eta)$ is sequentially determined from the boundary conditions obtained by substituting (2) in (5) as

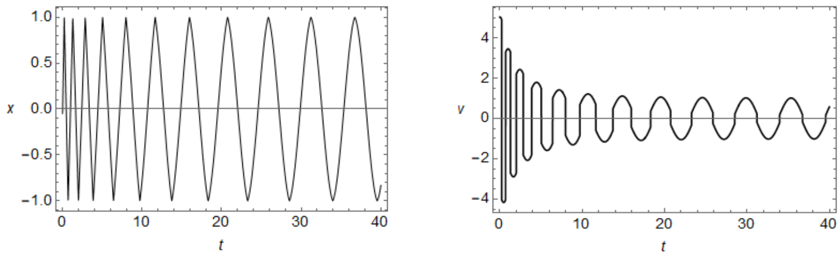
$$\tau = \pm 1: \quad Y = 0, \quad X = \pm\Delta, \quad X'\omega^2 = p \tag{6}$$

The condition on velocity in (4) gives

$$\tau = \pm 1: \quad Y' \mp X' = -(1 - \varepsilon)\left(Y' \pm X'\right) \tag{7}$$

As mentioned, conditions (6) eliminate singularities produced by substitution of (2) in (5), whereas conditions (7) require somewhat detailed consideration [4]. Then conducting two steps of the asymptotic procedure gives finally the closed form solution (Fig. 2)

$$\begin{gathered} x(t) = \Delta\left[\frac{\sin\lambda\tau}{\sin\lambda} - \frac{\varepsilon}{2\Omega}\frac{d\lambda}{d\eta}\left(\frac{\cos\lambda\tau}{\cos\lambda} - \tau\frac{\sin\lambda\tau}{\sin\lambda}\right)e\right] + O(\varepsilon^2) \\ \tau = \tau(\varphi), \quad \frac{d\varphi}{dt} = \frac{\Omega}{\lambda}, \quad \frac{d\lambda}{d\eta} = \frac{\Omega}{2}(1 + \cos 2\lambda)\left(1 + \frac{\sin 2\lambda}{2\lambda}\right)^{-1} \end{gathered} \tag{8}$$



(a) (b)

Figure 2: Time history of the state variables of impact oscillator with energy loss at its boundaries:
a) coordinate, and b) velocity showing transition to the so-called 'grazing' regime with near zero impact pulses.

Conclusion

A class of vibrating systems with perfectly stiff amplitude limiters is considered by means of non-smooth time substitutions. The motion is represented as a combination of the oscillating component, which is due to cyclic collisions with the limiters, and a slow decay caused by the energy loss at collision times. A specific modification of the two variable expansions is used, where the non-smooth (triangle wave) temporal argument is viewed as a fast time while the energy decay is described in a slow time scale. As a result, closed-form analytical solutions are obtained that automatically satisfy collision conditions with the energy loss. Three qualitatively different basic types of vibrations are considered to cover periodic, frequency modulated, and amplitude-frequency modulated motions.

References

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