Nonlinear dynamics of a resiliently propped cantilevered beam with a tip mass

M. Reza Talebi Bidhendi*, A. Srikantha Phani*
Department of Mechanical Engineering, University of British Columbia, Vancouver, Canada

Summary. A practical friction control application in railway industry lead us to consider a cantilever beam with a tip mass and a spring support– a problem least studied in the literature from nonlinear dynamics perspective. We study the harmonically forced nonlinear dynamic response of this prototypical system using a reduced two degree of freedom model and compare its bifurcation characteristics, computed by AUTO, with that of the full system. The tip spring couples the first two flexural modes through a length dependent kinematic nonlinearity and brings their frequency ratio to nearly 2:1. We find (a) the reduced order model is adequate to understand the first bifurcation (pitchfork) and the subsequent Hopf bifurcation and (b) The threshold force for the nonlinear response grows with the initial length of the spring.

Problem statement

Cantilevered beam with tip mass is a canonical problem extensively studied under the tip excitation and under the base excitation [1]. Nonetheless, the present problem of an end excited cantilevered beam with a tip mass mounted on an external elastic support has not received attention, and we are led to it through a friction control application in railways [2, 3]. Similar situations can arise in the vibration energy harvesting, vibration mitigation, and sensing mechanisms [4]. Restoring force due to the elastic support has been shown to produce nonlinear terms of kinematic origin in the governing differential equations (DEs) [5, 6]. A similar situation arises in our problem, sketched in Fig.1(a). A long steel beam with a diameter of 4 mm and length of 254 mm is loaded by an aluminum tip mass (70×50×20 mm^3) and a rigid rod of density of 7800 kg/m^3 and diameter of 4 mm and length of 40 mm. The external support at tip ($k_{ext} = 1000\ N/m$) with an initial length (L) imposes a two-to-one ratio between the first vertical (f_1) and the lateral (f_2) natural frequencies of the structure, $f_1 \approx 2f_2$. The linear modal parameters of the structure (damping ratio(ζ), stiffness(k), natural frequency (f)) without the addition of the external support are obtained using a finite element model as $k_1 = 317.98\ N/m, \zeta_1 = 0.097, f_1 = 6.12\ Hz, k_2 = 362.58\ N/m, \zeta_2 = 0.129, f_2 = 6.59\ Hz$. Indices one and two refer to the vertical and the lateral directions, respectively. After the attachment of the elastic support to the structure, the vertical linear natural frequency becomes $f_1 = 12.46\ Hz$ while the other linear modal parameters do not change. Using the linear modal properties of the structure, a minimal model as shown in Fig.1(b) can be identified, with the governing equations:

$$\ddot{\tilde{y}} + 2\zeta_1\omega_1\dot{\tilde{y}} + \omega_1^2\tilde{y} + \frac{k_{ext}(1+\tilde{y})}{m_1}(1 - \frac{1}{\sqrt{(1+\tilde{y})^2+\tilde{x}^2}}) = \frac{F}{m_1 L}cos(\Omega t) \quad (1)$$

$$\ddot{\tilde{x}} + 2\zeta_2\omega_2\dot{\tilde{x}} + \omega_2^2\tilde{x} + \frac{k_{ext}\tilde{x}}{m_2}(1 - \frac{1}{\sqrt{(1+\tilde{y})^2+\tilde{x}^2}}) = 0 \quad (2)$$

where $\tilde{y} = y/L$, $\tilde{x} = x/L$, $\omega_i = \sqrt{k_i/m_i}$; $m_i, i = 1, 2$ are the masses; F is the excitation amplitude, and Ω is the excitation frequency. The above equations are solved numerically for non-zero initial conditions, and compared with the full-scale multi-body dynamics model in ADAMS software which accounts for all the modes of the structure, i.e, no modal truncation.

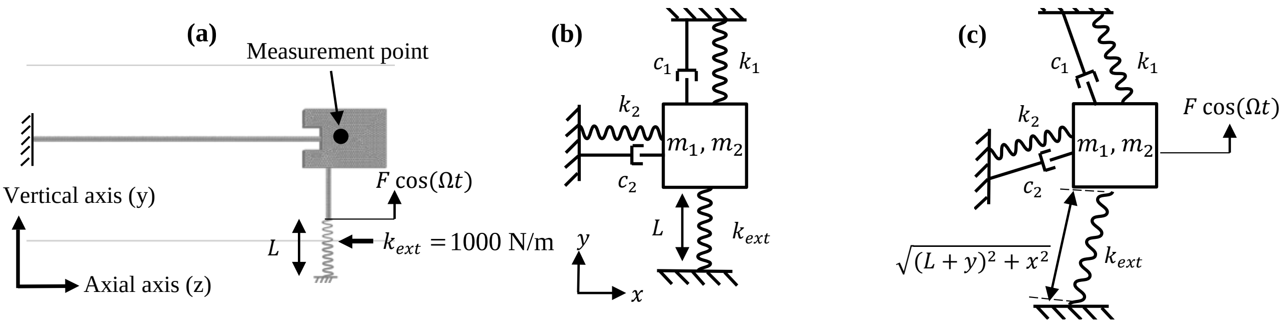

Figure 1: (a) Model of end excited cantilevered beam with tip mass mounted on the elastic support. (b) A minimal two-DOF model of the structure: $c_i = 4\pi\zeta_i m_i f_i$. (c) Deformed structure under the vertical excitation force. Note that the first vertical and lateral modes have almost the same natural frequencies ($f_1 \approx f_2$) without the external spring. The external support introduces kinematic nonlinearities and imposes 2:1 internal resonance in the structure ($f_1 \approx 2f_2$) . Note that x and y are the lateral and the vertical axis, respectively.

Results

Force-response curves are calculated by fixing the excitation frequency (Ω) and increasing the excitation amplitude (F), and vice versa [7]. When the excitation frequency is close to the first vertical natural frequency, $\Omega \approx 2\pi f_1$, the first lateral mode is excited through a 2:1 resonance (pitchfork bifurcation) as shown in Fig.2(a). The vertical displacement decreases initially, in Fig.2(b), and then increases. A further increase in F will produce a Hopf bifurcation leading to the aperiodic response of the structure, see the dotted line in Fig.2(a). Fixing the excitation amplitude and sweeping the excitation frequency, Fig.2(d) and Fig.2(e), we can observe that there is a frequency interval over which the lateral mode is activated. Internal resonance property is evident through the frequency splitting in Fig.2(f). The sensitivity of the force threshold to the initial length of the external spring is shown in Fig.2(c), where we observe that the force amplitude divided by the length remains constant, for the two lengths shown. Note that Fig.2(c) is a magnified version of Fig.2(a) near the onset of first bifurcation. We note that while the reduced order model is adequate for $L = 0.5\ mm$, discrepancies exist for $L = 2.5\ mm$, indicating the limit of the reduced model.

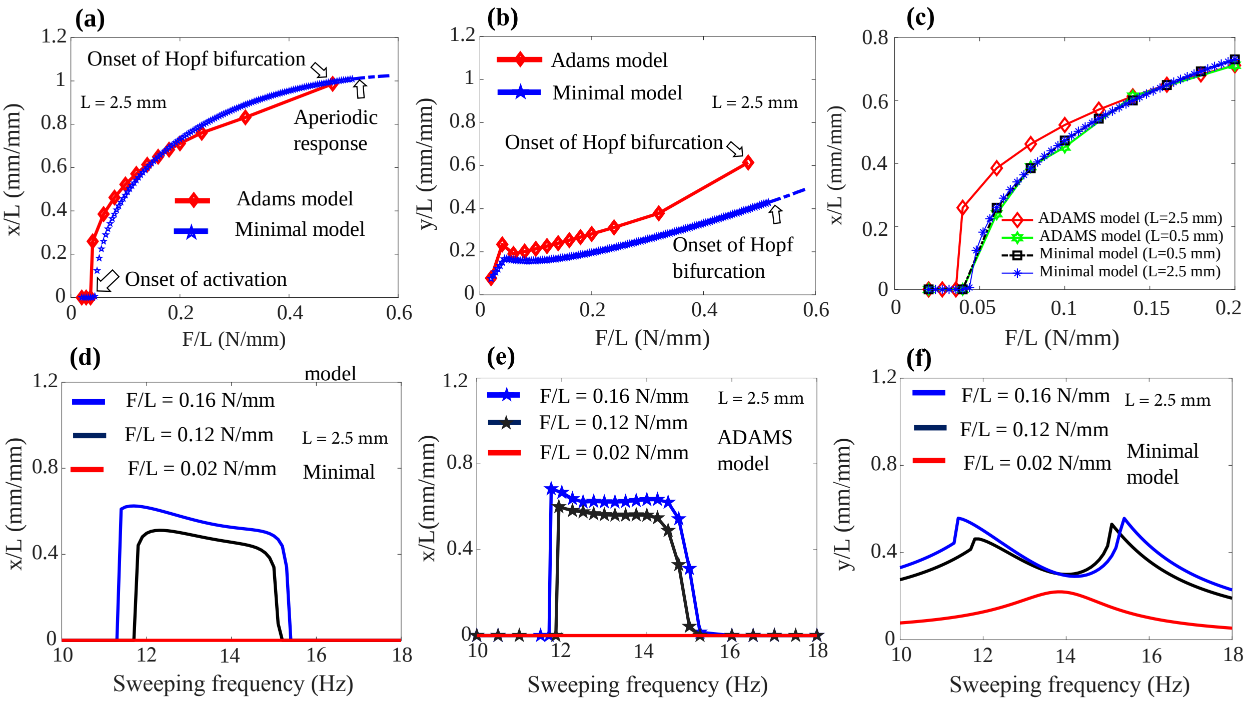

Figure 2: (a) Force-response curve of the lateral response with $\Omega = 2\pi \times 12Hz$. (b) Force-response curve of the vertical response with $\Omega = 2\pi \times 12Hz$. (c) Sensitivity of the force threshold to the initial length of the external spring. (d) Lateral response of the minimal model due to the different excitation levels and frequencies. (e) Lateral response of the Adams model for the different excitation levels and frequencies. (f) Vertical response of the minimal model due to the different excitation levels and frequencies.

Conclusions

The elastically propped cantilever beam with a tip mass, subjected to harmonic excitation, shows pitchfork and Hopf bifurcations when the forcing amplitude is increased at a fixed frequency, and vice versa. The 2:1 internal resonance between two flexural modes with displacements in mutually orthogonal planes are superficially similar to Froude oscillations of a ship, but without saturation since we retain all nonlinear terms here, albeit in the first two modes in our reduced model. We find (a) the reduced order model is adequate to understand the first bifurcation (pitchfork) and the subsequent Hopf bifurcation and (b) initial length of the spring acts as a design tuning parameter to activate the first lateral mode.

References

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