Free Vibration Analysis of Functionally Graded Plates with Crack or Slit by the R-Functions Method

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Summary. The proposed approach applies the Ritz method combined with the R-functions theory to analyze the free vibrations of cracked functionally graded material (FGM) plates of an arbitrary planform and different boundary conditions. Material properties are assumed to be temperature-independent and varying along the thickness direction according to Voigt's law. The paper proposes a new set of admissible functions in order to describe the effect of cracks on the frequencies and mode shapes of vibrations. The admissible functions constructed by the R-functions theory take into account discontinuous of deflection. A comparison of the obtained results with available ones for rectangular plates confirms validation of the proposed approach. The present method is employed to obtain the frequencies and mode shapes for FGM plates with complex form and internal cracks, having various locations and length. Effects of the gradient index, crack location, crack length and orientation on frequencies and mode shapes of FGM plates are studied.

Introduction

Free vibration analysis of cracked functionally graded materials (FGM) plates with complex planform is a difficult mathematical problem, but it is very important for investigation of the nonlinear dynamical behavior of many modern thin walled construction. Number of publications devoted to this problem is quite limited and concerns only rectangular plates [1-3]. In this study we propose to apply the R-functions theory to construct the corresponding admissible functions which take into account the discontinuous behavior of the deflection of FGM plates with complex planforms.
The developed approach is tested on linear vibration problems, what is important for solving nonlinear vibration problems.

Problem formulation

Assume that a plate is made from a mixture of ceramics (top of the plate) and metal (bottom). Below a power-law distributions (Voigt's model) of the volume fractions of the metal and ceramics is employed. Then the effective material properties $P_{ef}(z)$ such as Young's modulus E, Poisson's ratio ν, and mass density ρ can be expressed as [1-4]:

$$P_{ef}(z) = \left(P_c(T) - P_m(T)\right)\left(\frac{z}{h} + \frac{1}{2}\right)^k + P_m(T). \tag{1}$$

Mechanical characteristics of ceramic P_c and metal P_m depend on temperature T. This dependence is defined by known formula presented in work [4]. Stress and strain resultants in matrix form are as follows:

$$\{N\} = [A]\{\varepsilon^0\} + [B]\{\chi\}, \quad \{M\} = [B]\{\varepsilon^0\} + [D]\{\chi\}, \tag{2}$$

where $\{N\} = \{N_{11}, N_{22}, N_{12}\}^T$ are forces per unit edge length in the middle surface of a plate, $\{M\} = \{M_{11}, M_{22}, M_{12}\}^T$ are bending and twisting moments per unit edge length, whereas the components of the vectors $\{\varepsilon^0\} = \{\varepsilon^0_{11}, \varepsilon^0_{22}, \varepsilon^0_{12}\}^T$ and $\{\chi\} = \{\chi_{11}, \chi_{22}, \chi_{12}\}^T$ are defined in the following way

$$\varepsilon^0_{11} = u,_x \quad \varepsilon^0_{22} = v,_y \quad \varepsilon^0_{12} = u,_y + v,_x, \quad \varepsilon_{13} = \delta(w,_x + \psi_x), \quad \varepsilon_{23} = \delta(w,_y + \psi_y),$$
$$\chi_{11} = \delta\psi_x,_x - (1-\delta)w,_{xx}, \quad \chi_{22} = \delta\psi_y,_y - (1-\delta)w,_{yy}, \quad \chi_{12} = \delta(\psi_x,_y + \psi_y,_x) - 2(1-\delta)w,_{xy}. \tag{3}$$

where u, v are middle surface displacements along the axes Ox and Oy respectively, w is the transverse deflection of the plate along the axis Oz, ψ_x, ψ_y are angles of rotations of the normal to the middle surface about the axes Ox and Oy. Symbol $\delta = 1$ stands for first order shear deformation theory (FSDT) and $\delta = 0$ is associated with the classical theory (CLT). Elements of the matrixes $[A]$, $[B]$, $[D]$ have the following form:

$$([A],[B],[D]) \int_{-\frac{h}{2}}^{\frac{h}{2}} E(z)[C](1, z, z^2)dz, \quad [C] = \frac{1}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}. \tag{4}$$

The proposed method of investigation of free vibration of FGM plates is based on an application of the R-functions theory and the Ritz method [5, 6]. Variational statement of the vibration problem is reduced to determine the stationary point of the following functional

$$J = U - \omega^2 P, \tag{5}$$

where U and P are maximum potential and kinetic energies [3] relatively:

$$U = \frac{1}{2}\iint_{\Omega}\left(N_{11}\varepsilon_{11} + N_{22}\varepsilon_{22} + N_{12}\varepsilon_{12} + M_{11}\chi_{11} + M_{22}\chi_{22} + M_{12}\chi_{12}Q_x\varepsilon_{13} + Q_y\varepsilon_{23}\right)d\Omega\,, \tag{6}$$

$$P = \frac{1}{2}\iint_{\Omega} I_0\left(\dot{u},_t^2 + \dot{v},_t^2 + \dot{w},_t^2\right) + 2I_1\left(\dot{u},_t\,\dot{\psi}_x,_t + \dot{v}_t\dot{\psi}_y,_t\right) + I_2\left(\dot{\psi}_x,_t^2 + \dot{\psi}_y,_t^2\right)dxdy\,, \tag{7}$$

where

$$\left(I_0, I_1, I_2\right)\int_{-\frac{h}{2}}^{\frac{h}{2}}\rho(z)\left(1, z, z^2\right)dz\,. \tag{8}$$

Construction of admissible functions

Let us construct admissible functions for FGM plates clamped on part of the border and free on remain part of the border. Suppose that a crack or a slit is located on the straight line directed from the free plate part to inside domain. Equation of the clamped part of the border $\omega_1(x,y)=0$ can be constructed by the R-functions theory. Normalized equation of the crack or slit $\omega_2(x,y)=0$ can be also constructed using tools of the R-functions theory. If crack or slit lies on the line l with the normalized equation $l=0$ and the function $\varphi(x,y)\geq 0$ describes the area, separating the part of the line coinciding with length and position of the crack or slit, then the function $\omega_2(x,y)=0$ takes the following form

$$\omega_2(x,y) = \sqrt{l^2 \vee_0 \overline{\varphi}}\,, \tag{9}$$

where $\vee_0$ is a sign of the R-disjunction [6]. The R-function theory allows to construct an equation of the free part of the border $\omega_3(x,y)=0$. It is possible to prove that solution structure with an account of crack on free part of the boundary plate has the following form

$$W = \frac{\omega_1^2}{\omega_1^2 + \omega_3}\left(\Phi_1 q_1 + \Phi_2 q_2 + (1+\omega_3)\Phi_1\right), \tag{10}$$

where Φ_1, Φ_2, Φ_3 are indefinite components of the solution structure. Functions $q_1(x,y), q_2(x,y)$ are defined as

$$q_1(x,y) = \left(1 + D_1^{(l)}\omega_2(x,y)\right)/\,2, \quad q_2(x,y) = \left(1 - D_1^{(l)}\omega_2(x,y)\right)/\,2\,, \tag{11}$$

where $D_1^{(l)}\omega_2 = l,_x\,\omega_2,_x + l,_y\,\omega_2,_y$. It is possible to show that $D_1^{(l)}\omega_2 = \pm 1 + O(\omega_2)$.

The corresponding structures can be constructed for the functions u, v and $\psi_x\ \psi_y$ in a similar way. In order to get admissible functions, indefinite components Φ_1, Φ_2, Φ_3 are expanded into series over some complete system of functions $\left\{\varphi_i^{(k)}\right\}, k = 1,2,3.$

Conclusions

A new approach is proposed to study free vibrations of FG plates with cracks or slits. System of basic functions, which takes into account the shape of a crack and the discontinuous behavior of the deflection, is constructed by the R-functions theory. The proposed approach is validated against available results for the linear frequencies of cracked isotropic and FGM plates with various boundary conditions and rectangular planform. The comparison was made for simply supported square plate with center crack and cantilevered plate with a side crack. The obtained results are found to be in a good agreement with the known solutions. In order to demonstrate the possibilities of the proposed method and developed software, we have analyzed influence of the location, crack length on natural frequencies of the FGM plates with complex planform.

References

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