Free and forced modes of fractional-type torsional oscillations of a complex rod

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Summary. In this paper we study torsional oscillations of a complex rod made of material with viscoelastic properties. Free and forced modes of fractional-type torsional oscillations of a discrete system of a complex rod are described by a system of differential fractional order equations for a special case. The kinetic energy, deformation work, and generalized energy dissipation function of a fractional type system are identified for a special case. It has been shown that in the general case of such fractional type systems, with torsional oscillations, there are no independent modes, and that the system behaves as a nonlinear system.

Model of torsional oscillations of a complex rod

Figure 1 represents a model of a complex rod which is considered as a fractional-type discrete complex system. It consists of a discrete complex structure composed of light rigid rods of negligible mass, length ℓ_k , $k=1,2,3,4$ that bear material points m_k , $k=1,2,3,4$, at their free ends, and are rigidly connected to the main rod at an angle β_k , $k=1,2,3,4$ at the other end. The rods are in pairs symmetrically arranged and rigidly joined in the sections of the main rod 1,2,3 and 4 at distances $\frac{\ell}{4}$, $\frac{\ell}{2}$, $\frac{3\ell}{4}$ and ℓ, measured from the left fixed end of the main rod, and where ℓ is the length of the main rod. In the cross-section, at distance $\frac{\ell}{4}$ from left end of the main rod, a one-frequency momentum $\mathsf{M}_{z,1}=\mathsf{M}_{z,01}\sin(\Omega_1 t+\varphi_1)$, with amplitude $\mathsf{M}_{z,01}$, frequencies Ω_1 and φ_1 phase is applied.

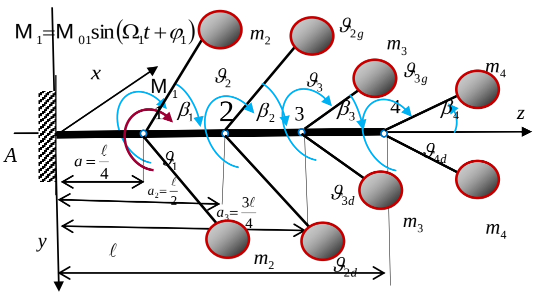

Figure 1. Model of a complex rod. This complex system which is considered as discrete system which torsional non-linear oscillations can be described using fractional derivatives.

For a case of an ideally elastic material, a segment rigidity of a torsion rod is equivalent to the rigidity of a torsion spring c_t and is determined by a constitutive relation of the torsion momentum-torsion angle, M_1 and ϑ. The rigidity of a torsion spring c_t is defined as $c_t=\frac{GI_0}{\ell}$ where G is the modulus of sliding of the rod's material, and $\mathbf{I}_0$ is the polar moment of the cross-sectional area of the main rod [1.2]. **Assumptions of the model:** all masses are equal and all angles are equal and the indentations are of equal lengths. The system is homogeneous. Materials of which beam and rods are made have viscoelastic properties and the constitutive relation of the moment M_1 of torsion of the angle ϑ, of torsion of the beam segment can be written in the form:

$$\mathsf{M}_t=-\left(c_t\vartheta+c_{t\alpha}\mathsf{D}_t^{\alpha}[\vartheta]\right) \tag{1}$$

where $c_{t\alpha}$ is the stiffness of the fractional type of dissipative properties of the torsion deformation of the beam segment considered as a torsion spring, and $\mathsf{D}_t^{\alpha}[\bullet]$ is a differential operator of the fractional order in the following form (see References [3-6]):

$$\mathsf{D}_t^{\alpha}[\vartheta(t)]=\frac{d^{\alpha}\vartheta(t)}{dt^{\alpha}}=\vartheta^{(\alpha)}(t)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t\frac{\vartheta(\tau)}{(t-\tau)^{\alpha}}d\tau\,,\quad \alpha\in(0.1),\ t\in(0.b) \tag{2}$$

where α is a rational number between 0 and 1, determined experimentally and $\Gamma(1-\alpha)$ is the Gamma function.

The kinetic energy, deformation work, and generalized energy dissipation function of the fractional type system

Free and forced modes of fractional-type torsional oscillations of a discrete system of a complex rod are described by a system of differential fractional order equations for a special case. The system of the fractional order differential equations can be written in matrix form:

$$\mathbf{A}\{\ddot{\vartheta}_k\}+\mathbf{C}\{\vartheta_k\}+\mathbf{C}_\alpha\left\{\mathsf{D}_t^\alpha[\vartheta_k]\right\}=0 \tag{3}$$

where $\mathbf{A}$ is the matrix of coefficients of inertia, $\mathbf{C}$ is the matrix of coefficients of elasticity and $\mathbf{C}_\alpha$ is a matrix of coefficients of the system of fractional properties. If the relation $c_{t,\alpha,k}=\kappa_\alpha c_{t,k}$, $k=1,2,3,4$ is valid, then $\mathbf{C}_\alpha=\kappa_\alpha\mathbf{C}$ and, in that case, the system belongs to a special class of fractional type systems in which main modes of fractional type are independent. In this case, we can apply a procedure for transformation of the generalized coordinates into the main coordinates like for the corresponding linear system.

Expressions for kinetic energy $\mathbf{E}_k$, deformation work $\mathbf{A}_{def}$ of the torsional deformed system, and the generalized function Φ_w [5,6] of fractional-type dissipation of the system energy are

$$\mathbf{E}_k=\frac{1}{2}\sum_{k=1}^{k=4}\mathbf{J}_{z,k}\dot{\vartheta}_k^2=\sum_{k=1}^{k=4}m_k\left(\ell_k\sin\beta_k\right)^2\dot{\vartheta}_k^2, \tag{4}$$

$$\mathbf{A}_{def}=\mathbf{E}_p=c_{t,1}\vartheta_1^2+\frac{1}{2}c_{t,2}(\vartheta_2-\vartheta_1)^2+\frac{1}{2}c_{t,3}(\vartheta_3-\vartheta_2)^2+\frac{1}{2}c_{t,4}(\vartheta_4-\vartheta_3)^2 \tag{5}$$

$$\Phi_w=\frac{1}{2}c_{t,\alpha,1}\left(\mathsf{D}_t^\alpha[\vartheta_1]\right)^2+\frac{1}{2}c_{t,\alpha,2}\left(\mathsf{D}_t^\alpha[\vartheta_2-\vartheta_1]\right)^2+\frac{1}{2}c_{t,\alpha,3}\left(\mathsf{D}_t^\alpha[\vartheta_3-\vartheta_2]\right)^2+\frac{1}{4}c_{t,\alpha,4}\left(\mathsf{D}_t^\alpha[\vartheta_4-\vartheta_3]\right)^2 \tag{6}$$

To obtain a system of fractional order differential equations that describe the torsional oscillations of the observed discrete complex structure, Lagrange differential equations of the second order were used**.**

Independent fractional order differential equations for describing free main fractional type modes are in the following form:

$$\ddot{\xi}_s+\omega_s^2\xi_s+\kappa_\alpha\omega_s^2\mathsf{D}_t^\alpha[\xi_s]=0,\ s=1,2,3,4 \tag{7}$$

and for describing forced main fractional type modes:

$$\ddot{\xi}_s+\omega_s^2\xi_s+\kappa_\alpha\omega_s^2\mathsf{D}_t^\alpha[\xi_s]=h_s\sin(\Omega_1 t+\varphi_1),\ s=1,2,3,4 \tag{8}$$

where ξ_s are main coordinates, ω_s are eigen frequencies of a fractional order system, κ_α is the constant of proportionality. When $c_{t,\alpha,k}\neq\kappa_\alpha c_{t,k}$ and $\mathbf{C}_\alpha\neq\kappa_\alpha\mathbf{C}$, the fractional-type modes are coupled and the system behaves in a non-linear way, and there is an interaction between fractional-type modes.

Conclusions

Torsional oscillations of a complex rod with viscoelastic properties are analyzed. The main idea applied in the paper is that the segment rigidity of the described torsion rod is equivalent to the rigidity of a torsion spring. The kinetic energy, deformation work, and generalized energy dissipation function of the fractional type system are identified. Free and forced main fractional type modes are defined. It has been shown that in the general case of such fractional type systems with torsional oscillations, there are no independent modes, and that the system behaves as a nonlinear system. When system belongs to an after-mentioned special class of fractional type systems, main modes of fractional type are independent.

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