# Destabilizing effect of back electromotive force along the cyclic coordinate in case of a digitally controlled Furuta pendulum

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*Summary*. Two usually neglected effects are considered in the mechanical model of the Furuta pendulum: the additional viscous damping caused by back electromotive force and the sampling delay caused by the digital control. These effects have relevant influence on the choice of the control algorithm and on the stability properties of the upward position of Furuta pendulum.

## Introduction

The Furuta pendulum [1] is a relatively simple two degree of freedom mechanical device (see the left panel of Figure 1), which has became a commonly used equipment for demonstrating various control algorithms applied on a highly nonlinear dynamical system. The arm is usually driven by an electric DC motor and the pendulum hangs freely, so this system is underactuated. The contol signal is the input voltage of the motor, and its calculation by means of a digital controller requires the consideration of both the sampling delay and the additional viscous damping caused by the back electromotive force appearing in the real system. This work considers these two often neglected effects; the consequences on stability and possible control algorithms are discussed.

## Governing equations

The nonlinear equations of motion of the Furuta pendulum can be obtained by means of the Lagrangian equations of the second kind in the form of:

$$(J_\mathrm{a} + J_\mathrm{p}\sin^2\theta)\ddot{\varphi} - (m_2 rl\cos\theta)\,\ddot{\theta} + 2J_\mathrm{p}\,(\sin\theta\cos\theta)\,\dot{\theta}\dot{\varphi} + (m_2 rl\sin\theta)\,\dot{\theta}^2 = M\,, \tag{1}$$

$$J_\mathrm{p}\ddot{\theta} - (m_2 rl\cos\theta)\,\ddot{\varphi} - J_\mathrm{p}\,(\sin\theta\cos\theta)\,\dot{\varphi}^2 - m_2 gl\sin\theta = 0\,, \tag{2}$$

where $\theta$ is the pendulum angle, $\varphi$ is the arm angle, $J_\mathrm{p} = m_2 l^2 + J_2$ and $J_\mathrm{a} = m_1(r/2)^2 + m_2 r^2 + J_1$ are corresponding mass moments of inertia; the other mechanical parameters are shown in Figure 1. The control torque $M$ applied on the arm is usually provided by a DC motor, for which the output torque can be determined based on the governing equations of the electric motor as a function of the input voltage $U_\mathrm{in}$ and the motor angular speed which is proportional to the arm angular velocity $\dot{\varphi}$ of the Furuta pendulum:

$$M = NU_\mathrm{in} - K\dot{\varphi}. \tag{3}$$

The constants $N$ and $K$ are the motor parameters related to the input voltage and back electromotive force, respectively. Note that substituting this into the equation of motion, the back electromotive force is analogous to a viscous damping force applied at the arm. Considering that the input voltage is determined by a digital microcontroller, sampling delay appears in the feedback loop in the following form

$$U_\mathrm{in}(t) = -P_1\theta(t_{j-1}) - D_1\dot{\theta}(t_{j-1}) + D_2\dot{\varphi}(t_{j-1}), \quad t \in [t_j, t_{j+1}), \tag{4}$$

where $P_1, D_1, D_2$ are the control gains; $t_j = j\tau$, $j \in \mathbb{Z}$ is the $j^\mathrm{th}$ sampling instant, and $\tau$ is the sampling time, which is assumed to be constant. Alternatively, if the angular velocity $\dot{\varphi}$ cannot be measured, then the application of an integral term can be considered in the feedback loop:

$$U_\mathrm{in}(t) = -P_1\theta(t_{j-1}) - D_1\dot{\theta}(t_{j-1}) - I_1\tau\sum_{i=0}^{j-1}\theta(t_i), \quad t \in [t_j, t_{j+1}), \tag{5}$$

In what follows, we discuss the case of Equation (4).

## Results

The sampling delay causes stability problems in the control systems as shown in [3, 4]. Based on the linearized system model, the critical sampling time can be calculated; this is the maximal sampling time of the digital control for which the pendulum can be stabilized with appropriate control gains. The minimal value of $D_{2,\min} > 0$ control gain can also determined, which means the simplest $\mathrm{P_1D_1}$ controller (with $D_2 = 0$) cannot stabilize the system. In other words, the upward pendulum position $\theta = 0$ is unstable for any $P_1, D_1$ control gains without feedback of the arm angular velocity (or the integral of the pendulum angle, see (5)). This instability is caused by the back electromotive force of the DC motor, which is similar to the viscous damping along the cyclic coordinate, while the inverted pendulum is underactuated

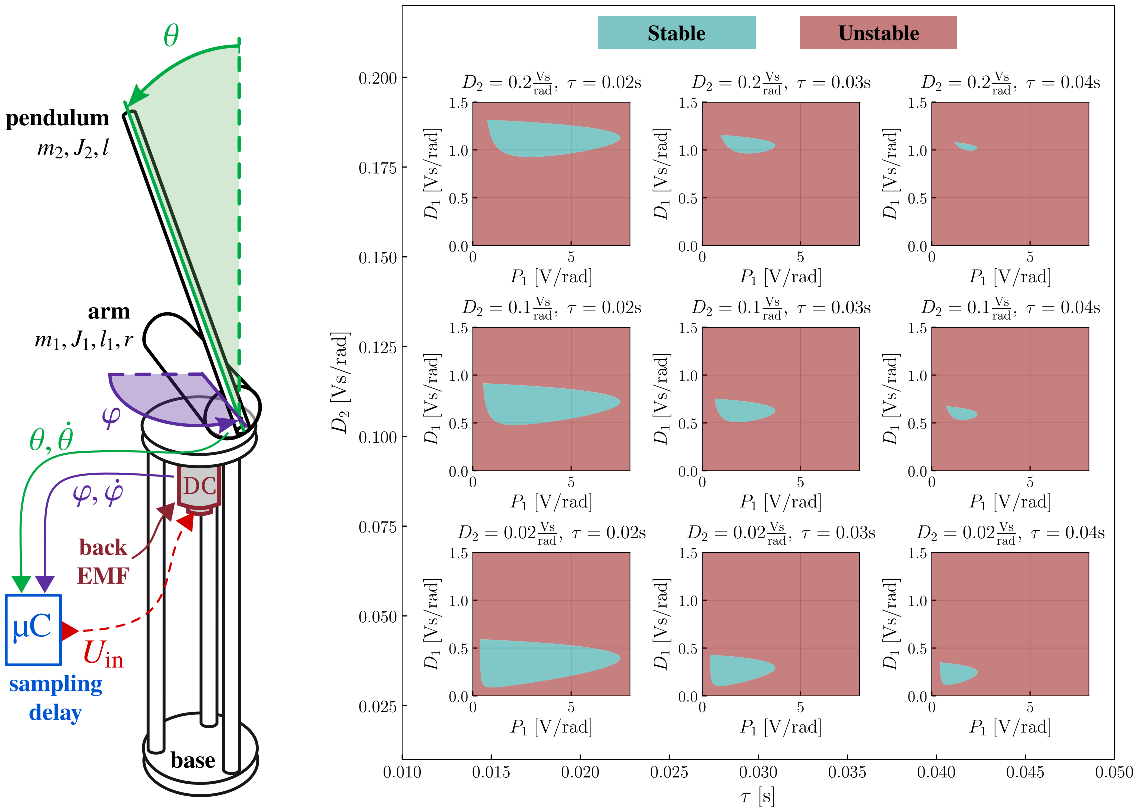


Figure 1: The Furuta pendulum (left) and the four dimensional stability chart of the Furuta pendulum which takes into account the back electromotive force of the DC motor and the sampling delay of the feedback loop (right).

due to the control force applied along the cyclic coordinate only. In case of standard positioning tasks of fully actuated systems, the back electromotive force does not cause such kind of instability, moreover, it is often considered useful because of its extra damping effect. In case of the underactuated control task like the Furuta pendulum, the presence of back electromotive force makes the system inherently unstable in case of classical PD control, this is the reason why an improved control law is needed to achieve stable system behavior: either the feedback of the arm (motor) angular velocity $\dot{\varphi}$ with a $D_2 > D_{2,\text{min}}$ control gain is needed, or an integral gain has to be used with respect to the pendulum angle $\theta$. The results of the corresponding calculations are represented by stability charts in the right panel of Figure 1, in the space of four parameters: the sampling time $\tau$ and the three control gains $P_1, D_1, D_2$. It can be seen, that the size of the stable parameter region becomes smaller with increasing sampling time $\tau$ and it completely vanishes at the identified critical values. The effect of the parameter $D_2$ is similar: the stable regions disappear above maximal and below minimal gain values. The stability charts are similar if the integral gain $I_1$ is applied as shown in Equation (5).

## Conclusions

It was found that the simple $\mathrm{P_1D_1}$ control law applied on the pendulum angle $\theta$ and angular speed $\dot{\theta}$ is insufficient in the presence of cyclic viscous damping or back electromotive force; an improved control algorithm is necessary, by extending the $\mathrm{P_1D_1}$ controller with an extra derivative term $D_2$ applied on the angular speed $\dot{\varphi}$ or with an extra integral term $I_1$ applied on the pendulum angular position $\theta$. The critical sampling times of the digital control loops can also be determined, which are identified also by laboratory experiments on the Furuta pendulum.

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