Model reduction for hyperbolic systems with application to managed pressure drilling

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Summary. This abstract presents a model reduction approach for systems of hyperbolic partial differential equations (PDEs) with nonlinear boundary conditions for drilling applications. These systems can be decomposed into a feedback interconnection of a linear hyperbolic PDE subsystem and a static nonlinear mapping. We show that the linear PDE subsystem can effectively be approximated by a cascaded system of continuous time difference equations (CTDEs) and ordinary differential equations (ODEs). The performance of the proposed technique is investigated by application to a model for managed pressure drilling (MPD).

Introduction

Hyperbolic partial differential equations (PDEs) govern a variety of physical phenomena such as fluid mechanics. These models have in recent years gained much attention in the design of model-based control systems [1]. However, the complexity of these models currently handicaps the design of controllers that can meet advanced performance criteria. To enable controller synthesis for such performance criteria, it is common practice to approximate the hyperbolic system by finite-dimensional models in terms of ordinary differential equations (ODEs). However, this type of approximation exhibits a low accuracy in capturing the wave propagation effect in hyperbolic systems. By contrast, it is known that the boundary behaviour of hyperbolic systems without coupling source terms can exactly be described by low-order continuous-time difference equations (CTDEs). These facts motivate us to investigate a combination of ODEs and CTDEs for the approximating hyperbolic models.

In this study, we consider a special class of systems consisting of two sets of linear isothermal Euler equations with coupling source terms and nonlinear boundary conditions. This particular class of hydraulic models is used in a variety of engineering applications, such as in managed pressure drilling (MPD) automation [2]. We show that these models can effectively be approximated by a series connection of low-order models in terms of CTDEs and ODEs. The CTDE part is employed to embed the advective nature of the system and the ODE part is used to approximate in-domain coupling effects between system variables due to source terms. Because the wave propagation effect is already captured through the CTDE model, the ODE part no longer needs to be of high order. Finally, we apply the proposed model reduction technique to single-phase flow MPD-controlled drilling systems, and present simulation results to illustrate the effectiveness of this method for such MPD applications.

Problem statement

Consider a system of balance laws, consisting of two isothermal Euler equations

$$\frac{\partial Q}{\partial t} + \Psi_c \frac{\partial Q}{\partial \xi} + F_c Q = 0, \quad \Psi_c = \begin{bmatrix} \Psi & 0 \\ 0 & \Psi \end{bmatrix}, \Psi = \begin{bmatrix} 0 & 1 \\ c^2 & 0 \end{bmatrix}, F_c = \begin{bmatrix} F_1 & 0 \\ 0 & F_2 \end{bmatrix}, \tag{1}$$

where $\xi \in [0, l]$ and $t \geq 0$ are the spatial and temporal variables, $Q(t, \xi) \in \mathbb{R}^4$ is the vector of variables, and c [m/s] and l [m] are the speed of sound and length of the spatial domain, respectively. Moreover, $F_1, F_2 \in \mathbb{R}^{2\times 2}$ characterise the source terms. We assume a zero initial condition $Q(0, \xi) = 0$ and consider boundary conditions of the form

$$\Pi_1 \begin{bmatrix} Q(t,0) \\ Q(t,l) \end{bmatrix} - \Pi_2 \psi\left(\Gamma \begin{bmatrix} Q(t,0) \\ Q(t,l) \end{bmatrix}, u(t)\right) = 0, \quad \Pi_1 \in \mathbb{R}^{4\times 8}, \Pi_2 \in \mathbb{R}^{4\times n_l}, \ \Gamma \in \mathbb{R}^{r\times 8}, \tag{2}$$

where $u(t) \in \mathbb{R}^p$ is the input vector and the nonlinear function $\psi(\cdot,\cdot)$ is due to nonlinearities in the boundary conditions. Furthermore, we assume that the output is given by $y(t) = H\Gamma[Q^T(t,0), Q^T(t,l)]^T$, with $H \in \mathbb{R}^{m\times r}$.

Given the model in (1) and (2), the objective is to approximate the input-output behaviour of this system from the input u to the output y with a model of lower complexity, which allows for faster yet accurate time-domain simulations. Moreover, this model should possess a structure that potentially facilitates the design of high-performance controllers.

The system described by (1) and (2) can be cast into a feedback interconnection of an infinite-dimensional linear system Σ and a static nonlinear mapping $\psi(\cdot,\cdot)$ of the following forms:

$$\Sigma : \begin{cases} \dfrac{\partial Q}{\partial t} + \Psi_c \dfrac{\partial Q}{\partial \xi} + F_c Q = 0, Q(\xi, 0) = 0, \\ \Pi_1 \begin{bmatrix} Q(t,0) \\ Q(t,l) \end{bmatrix} = \Pi_2 v(t), w(t) = \Gamma \begin{bmatrix} Q(t,0) \\ Q(t,l) \end{bmatrix}, y(t) = Hw(t), \end{cases} \tag{3}$$

$$v(t) = \psi(w(t), u(t)), \tag{4}$$

where $w(t) \in \mathbb{R}^r$ is the output of Σ and $v(t) \in \mathbb{R}^{n_l}$ is its input. This decomposition enables us to reduce the complexity of this model by only reducing the complexity of the linear PDE part and leave the nonlinearities untouched.

Model reduction

In the absence of source terms, Σ can be modelled by a system of CTDEs, which represent the transport phenomenon in the system. However, the source terms lead to distributed in-domain couplings among the travelling waves. Our observations show that these interactions in particular affect the low-frequency behaviour of the system Σ. We can also show that the transfer function of Σ converges to a periodic behaviour (in terms of the frequency variable) of a period of $2\pi c/l$ at high frequencies. This periodic behaviour in the transfer function is a manifestation of the advective nature of the system. Thus, we conclude that in the presence of these source terms, the system behaviour is composed of two dominating aspects: 1) advection and 2) dynamics governing the shape of advective waves. As said before, the (advection-induced) transport aspects can be modelled by CTDEs. This is the dominating aspect at high frequencies. Given that the second aspect is mostly dominant at low frequencies, we compensate for that by a system of ODEs. This explanation motivates us to consider for $\hat{\Sigma}$, the approximate of Σ, a structure which consists of an interconnection of a CTDE model Σ_c and an ODE model Σ_o. Here, we adapt a series interconnection between Σ_c and Σ_o, and refer to it as the cascaded system. We propose the following realizations for Σ_c and Σ_o:

$$\Sigma_c : \begin{cases} E_1 x_1(t) = -A_1 x_1(t-\tau) + B_1 \hat{v}(t), \\ \quad z(t) = C_1 x_1(t), \end{cases} \quad , \quad \Sigma_o : \begin{cases} E_2 \dot{x}_2(t) = A_2 x_2(t) + B_2 z(t), \\ \quad \hat{w}(t) = C_2 x_2(t) + D_2 z(t), \end{cases} \tag{5}$$

where $x_1(t) \in \mathbb{R}^{n_1}$ and $x_2(t) \in \mathbb{R}^{n_2}$ are the state vectors, $\hat{v}(t)$ and $\hat{w}(t)$ are approximates of $v(t)$ and $w(t)$, respectively, and $z(t) \in \mathbb{R}^m$ is the output of Σ_c and the input to Σ_o. Moreover, $\tau = l/c$ is the delay and E_1, E_2, A_1, A_2, C_1, C_2, B_1, B_2 and D_2 are system matrices of appropriate dimensions. To construct Σ_c and Σ_o, the data-based method in [3] is used. Namely, Σ_c is designed such that its transfer function matches the transfer function of Σ at interpolation points chosen at high frequencies. By contrast, the ODE part Σ_o is designed such that it compensates for the error between Σ and Σ_c at interpolation points in the low-frequency range.

Simulations

In this section, we apply the presented model reduction technique to a model for MPD [2]. Fig. 1 shows the response of the downhole pressure in the annulus to step changes in the input u, consisting of the choke opening and pump flow. As can be observed, the nonlinear cascaded approximation yields an accurate approximation of the original system response, obtained from a discretization, especially in capturing the staircase pressure profile induced by wave propagation effects characteristic to the PDE model.

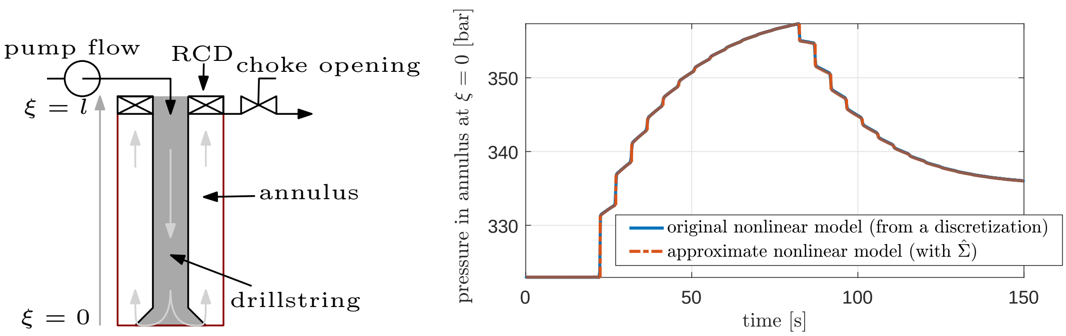

Figure 1: (Left) A simplified schematic of a drilling system with MPD equipment, (right) comparison between the time-domain response of the reduced nonlinear model and the original model for the bottom-hole pressure (pressure in the annulus at $\xi = 0$).

Conclusions

This abstract presented a model reduction technique for systems of hyperbolic partial differential equations with nonlinear boundary conditions. The reduction is achieved by, first, decomposing the model into a feedback interconnection of a linear infinite-dimensional subsystem and a nonlinear mapping and, second, approximating the linear part by a series connection of a system of delay difference equations and a system of ordinary differential equations. The high accuracy of the approximation has been verified by application to a model for managed pressure drilling.

References

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