Theoretical considerations of the mechanics of whisker sensors

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Summary. Employing elastic rod theory we study the question which forces and moments measured at the base of a mammal's whisker (tactile sensor) allow for the prediction of the location in 3D space of the point at which the whisker makes contact with an object. We show that, in the case of non-tip contact, the minimum number of independent forces or moments is three but that conserved quantities of the rod equilibrium equations prevent certain triples from giving a unique solution. The existence of these conserved quantities depends on the shape and material properties of the whisker. For tapered or intrinsically curved whiskers there is no obstruction to the prediction of the contact point. Our results explain recent numerical observations in the literature and offer guidance for the design of robotic tactile sensory devices.

Introduction

Mammal whiskers (vibrissae) allow terrestrial animals to obtain information about geometrical and mechanical properties of the environment [1]. Animal whiskers are thin flexible rods grown out of follicles and consist of dead cells; there are no sensors along the length of a whisker. Sensing therefore relies on the detection by mechanoreceptors at the whisker base of forces and moments induced by contact with an external object and transmitted through the elastic medium. Knowledge of how whiskers perform their sensory functions is of interest to engineers designing artificial tactile sensors [2]. To use such artificial whiskers in robotics, it is essential to be able to determine the location, with respect to a reference frame, of the point along the whisker shaft at which contact with an object occurs.

If three forces and three moments (in three independent spatial directions) are measured at the (fixed) whisker base, then a suitable mechanical model of the whisker (e.g., a 1D continuum elastic rod or beam model [3]) allows the entire configuration of the whisker, and hence the contact point, to be determined. These six measurements, however, require an expensive six-axis load cell. It is natural, therefore, to ask whether fewer measurements would suffice to uniquely predict the location of the contact point.

Past studies of this contact problem have mainly focussed on the planar case, where the contact point is specified by two coordinates [4]. Whisker configurations, especially those with intrinsic curvature, may generally be non-planar. The whisker contact problem was numerically studied in 3D in [5]. All 20 possible combinations of triples of base forces and moments were analysed, however no theoretical explanation of the results was given.. Here we show that the difference in the predictive ability of triples of forces and moments is mainly caused by the existence of conserved quantities, which arise for whiskers with certain geometrical profiles (curvature, taper, etc.).

Boundary conditions and conserved quantities

The solution of an nth-order ODE, $du/ds = f(u)$, $u \in \mathbb{R}^n$, $s \in [a, b]$, involves n integration constants. In physical problems a unique solution is then usually obtained by imposing n boundary conditions at $s = a$ and/or $s = b$ to fix those integration constants. For a linear ODE it is a rigorous result that a unique solution is obtained if the n boundary conditions are linearly independent. For a nonlinear ODE (or nonlinear boundary conditions) the result is only true locally (i.e., near a given solution) and only 'generically', i.e., away from branching points (bifurcations) for special values of any parameters in the equations or values imposed at the boundary. (In the special case that all boundary conditions are specified at one end, i.e., for an initial-value problem, a unique solution is guaranteed also for a nonlinear ODE.)

A conserved quantity (first integral) of the ODE is a function of the dependent variables u_i $(i = 1, ..., n)$ whose value is constant along solutions of the equation. The presence of such quantities may put constraints on the specification of boundary conditions [6]. For instance, in the simple case that one of the variables itself, say u_k, is a conserved quantity and we choose the boundary condition $u_k(a) = c$, then $u_k(b) = d$ is not a proper boundary condition at the other end: if c and d were unequal there would obviously be no solution, while if c and d were equal there might be infinitely many solutions (depending on the other boundary conditions). In either case the BVP is said to be ill-posed. Another, independent, boundary condition needs to be imposed instead to obtain a locally unique solution (i.e., a solution with no infinitesimally close solutions). It is not always a priori clear that a given ODE has one or several conserved quantities and well-posedness of a nonlinear BVP is generally not straightforward.

Conserved quantities can be viewed as continuous symmetry properties of the ODE. A more obvious example of continuous symmetry is rotational symmetry of the equations, in which case for a well-posed BVP one has to impose boundary conditions that break the symmetry, thereby picking out one of the continuous family of solutions. Besides continuous symmetry a BVP may also have discrete symmetry, for instance, reflection symmetry, in which case the BVP has multiple *isolated* solutions. Each of these will generally be locally unique and the BVP is considered well-posed, with the solution being *globally* non-unique. 'Modes' (i.e., eigenfunctions) in eigenvalue problems, which also occur as isolated solutions, are other examples of globally non-unique solutions. Conserved quantities do not necessarily distinguish between such solutions.

Equilibrium equations for an elastic whisker

We model the whisker-object contact problem by formulating a two-point boundary-value problem using Kirchhoff rod theory [3]. Let $Oxyz$ be an orthogonal laboratory frame fixed at the base of the whisker (Fig. 1). The whisker is taken to be inextensible and unshearable and to have length L. Its centreline is denoted by $\mathbf{r}(s) = (x(s), y(s), z(s))$, where $s \in [0, L]$ is arclength along the whisker, $s = 0$ corresponding to the base O and $s = L$ corresponding to the tip. Under the above assumptions we can introduce an orthonormal material frame $\{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$ with $\mathbf{d}_1$ tangent to the centreline $\mathbf{r}$, i.e., $\mathbf{r}' = \mathbf{d}_1$, and $\mathbf{d}_2$ and $\mathbf{d}_3$ directed along principal axes of the whisker's cross-section (here and in the following a prime denotes differentiation with respect to s). By orthonormality of the material frame there exists a vector $\mathbf{\Omega}$ such that $\mathbf{d}_i' = \mathbf{\Omega} \times \mathbf{d}_i \quad (i = 1, 2, 3)$. The components of this vector in the material frame, $(\omega_1, \omega_2, \omega_3) =: \boldsymbol{\omega}$, $\omega_i = \mathbf{\Omega} \cdot \mathbf{d}_i$, are the strains of the theory, i.e., the curvatures ω_2 and ω_3, about $\mathbf{d}_2$ and $\mathbf{d}_3$, and the twist ω_1, about $\mathbf{d}_1$ [3].

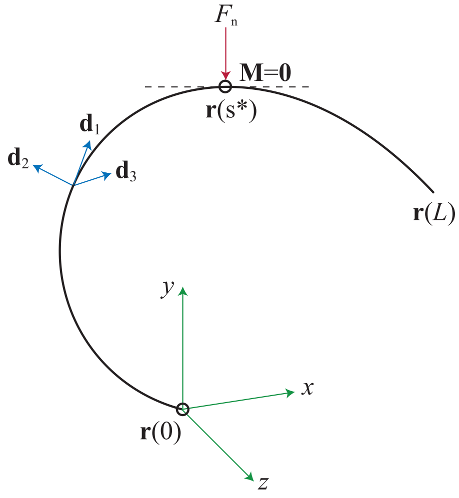

Figure 1: Coordinate systems for a whisker in point contact with an object at $s = s^*$. For frictionless contact the contact force F_n is normal to the whisker.

The force and moment balance equations for the whisker are

$$\mathbf{F}' + \boldsymbol{\omega} \times \mathbf{F} = \mathbf{0}, \quad \mathbf{M}' + \boldsymbol{\omega} \times \mathbf{M} + \mathbf{i} \times \mathbf{F} = \mathbf{0}, \tag{1}$$

where $\mathbf{F} = (F_1, F_2, F_3)$ and $\mathbf{M} = (M_1, M_2, M_3)$ are triples of force and moment components in the material frame and $\mathbf{i} = (1, 0, 0)$. We assume the linear constitutive relations: $M_1 = C(s)\omega_1, \quad M_2 = B(s)\omega_2, \quad M_3 = B(s)(\omega_3 - \omega_{30}(s))$. Here, $B(s)$ and $C(s)$ are the bending and torsional stiffnesses, resp. They are not constant in the particular case of a tapered rod. The undeformed shape of the whisker is assumed planar but may be curved with intrinsic curvature $\omega_{30}(s)$.

Eqs. (1) imply, respectively, that $\mathbf{F} \cdot \mathbf{F}$ and $\mathbf{F} \cdot \mathbf{M}$ are constant. If $\omega_{30} \equiv 0$, then $M_1 = \text{const}$. The Hamiltonian $H = M_1^2/(2C) + (M_2^2 + M_3^2)/(2B) + M_3\omega_{30} + F_1$ is conserved provided B, C and ω_{30} are constant, i.e. the rod is translationally symmetric in the arclength s [6].

Table 1: Triples $\mathbf{P}$ of measurements that give rise to an **ill-posed** BVP with non-isolated solutions, for various intrinsic shapes of the rod (* stands for any of the other quantities).

rod	cylindrical	tapered
straight ($\omega_{30} = 0$)	$(M_1, *, *)$ $(\alpha, \beta, *)$	
	(F_1, M_2, M_3) $(F_1, M_n, *)$	
curved ($\omega_{30} \neq 0$)	-	-

We assume the whisker to be fixed in both position and orientation at the base ($s = 0$). At the contact point ($s = s^\star \leq L$) a normal contact force will act from the surface of the object onto the whisker for a frictionless single-point contact. We therefore consider the following boundary conditions: $\mathbf{r}(0) = \mathbf{0}, \mathbf{d}_i(0) = \mathbf{d}_{i,0}, \mathbf{P}(0) = \mathbf{P}_0, F_1(s^\star) = 0, \mathbf{M}(s^\star) = \mathbf{0}$, where $\mathbf{P} = (P_1, P_2, P_3)$ is the vector of base measurements consisting of three components chosen from the six force and moment components. We also introduce polar representations of the force and moment in the whisker [5]. Thus we write $F_2 = F_n \cos\alpha$, $F_3 = F_n \sin\alpha$, $M_2 = M_n \cos\beta$, $M_3 = M_n \sin\beta$. Here $F_n = \sqrt{F_2^2 + F_3^2}$ and $M_n = \sqrt{M_2^2 + M_3^2}$ are the magnitudes of the normal force and moment components, while α and β are the angles these components make with the material axes.

We note that three base measurements are sufficient for the well-posedness of the tactile sensing BVP. We analyse how this well-posedness depends on the precise choice of measured components P_i.

Summary of results

The results of our analysis are summarised in Table 1, where those combinations of force/moment measurements are listed that are **not appropriate** in the design of an effective set of sensors at the base of a robotic whisker. A tapered whisker may here be interpreted as any whisker whose B or C is not constant.

References

[1] Evans M. H., Loft M. S. E., Campagner D. and Petersen R. S. (2019) Sensing the Environment with Whiskers. Oxford Research Encyclopedia, Neuroscience (oxfordre.com/neuroscience), subject: Sensory Systems.

[2] Pipe T. and Pearson M. J. (2016) Whiskered Robots. Scholarpedia of Touch, Scholarpedia.

[3] Love A. E. H. (1927) A Treatise on the Mathematical Theory of Elasticity. Cambridge University Press.

[4] Solomon J. H. and Hartmann M. J. Z. (2011) Radial distance determination in the rat vibrissal system and the effects of Weber's law. *Philosophical Transactions of the Royal Society B: Biological Sciences*, **366**(1581):3049–3057.

[5] Huet L. A., Rudnicki J. W. and Hartmann M. J. Z. (2017) Tactile sensing with whiskers of various shapes: Determining the three-dimensional location of object contact based on mechanical signals at the whisker base. *Soft Robotics*, **4**(2):88–102.

[6] Dichmann D. J., Li, Y. and Maddocks, J. H. (1996) In: Mathematical Approaches to Biomolecular Structure and Dynamics, volume 82 of IMA Volumes in Mathematics and its Applications, chapter Hamiltonian formulations and symmetries in rod mechanics, pages 71 – 113. The Institute for Mathematics and Its Applications.