Leveraging Rotating Frame Dynamics for Low-Power Chaos Generation in Nonlinear M/NEMS Resonators

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Summary. This work presents an approach to chaos generation that both preserves and leverages the rotating frame approximation that is usually used to analyse nonlinear M/NEMS resonators. The approach relies on increasing the phase-space dimensions to meet the Poincaré-Bendixon condition. Chaos generation is further constrained within the parameter space by relying on arguments from Melnikov's method. Experimental validation is performed using a GaAs piezoelectric nonlinear MEMS resonator.

Introduction and Theory

The relatively low-loss and weakly nonlinear properties of M/NEMS resonators have made the use of perturbation based techniques to capture their dynamics a marking feature of this field of research. These perturbation techniques include multiple scale analysis and the rotating frame approximation, and assume that the dynamics take the form of a "slow-flow" envelope superimposed on an otherwise sinusoidal carrier. These same physical properties that make M/NEMS devices desirable and perturbations techniques possible are usually counterproductive when it comes to generating chaos, since this latter is not a perturbation phenomenon.
Mathematically this effect can be understood by looking at the governing Duffing equation, written in a non-dimensional form as:

$$\ddot{x} + \gamma\dot{x} + x + \alpha x^3 = F_1 cos(\omega_1 t) + F_2 cos(\omega_2 t) \tag{1}$$

where x is the displacement, γ and α correspond to the non-dimensional dissipation and nonlinearity terms, and F_1, F_2, ω_1 and ω_2 are the magnitudes and frequency of externally applied forcing terms. Note that for the rotating frame approximation to apply, the frequency difference between $(\omega_1 - \omega_2) \ll 1$
The application of the rotating frame approximation to equation (1) gives:

$$\begin{aligned} \dot{X} &= -\delta Y + \frac{3\alpha}{8}(X^2+Y^2)Y - \frac{1}{2}(F_2 sin(\Theta) + \gamma X) \\ \dot{Y} &= \delta X - \frac{3\alpha}{8}(X^2+Y^2)X + \frac{1}{2}(F_1 + F_2 cos(\Theta) - \gamma Y) \\ \dot{\Theta} &= \Omega = (\omega_2 - \omega_1)/\omega_0 \end{aligned} \tag{2}$$

where, $\delta = (\omega_1 - \omega_0)/\omega_0$, and X and Y are the rotating-frame quadratures.
Equation (2) explains why it is difficult to generate chaos within the range of applicability of the rotating frame approximation in the case of only one forcing term is applied, i.e. $F_2 = 0$. Since in such a case the system reduces to a two-dimensional system ($n = 2$), and does not possess the necessary dimensions, i.e. $n = 3$, for chaos generation.
The typical approach to chaos generation in nonlinear M/NEMS devices have been to operate the device beyond the regime where equation (2) is valid, usually by applying large driving amplitudes and using exotic nonlinearity [1, 2]. However, it is equally possible to generate chaos while remaining within the perturbation regime, by expanding the rotating frame dimensions from $n = 2$ to $n = 3$, which can be done by applying the second tone, i.e. $F_2 \neq 0$.
While increasing the rotating-frame dimensions from 2 to 3 is a necessary condition, it nevertheless does not define the area within the four-dimensional parameter-space (δ, Ω, F_1, F_2) where chaos can exist. Fortunately, the system of equation (2) is typically described by Melnikov's method [3], which imposes the existence of a homoclinic bifurcation as a precondition for chaos generation. The existence of a homoclinic orbit implies operating within the bistable regime. Thus, it is an equally necessary condition to have at least one of the two applied tones within the bistable area of operation shown in Fig. 1(a)-(c) as a function of the non-dimensional parameters.

Experimental Validation

Experimental validation is performed using a GaAs piezoelectric MEMS clamped-clamped beam resonator. Under low drive amplitudes (70 mV), the single-tone frequency response of the device shows a resonance frequency and quality factor of $f_0 = 1.559$ MHz and $Q = 1000$ respectively, Fig. 1(d). For large amplitude sweeps (2.8 V), the device exhibits a hardening-type Duffing nonlinearity, which is fitted to give an $\alpha = 16$, Fig. 1(d).
Upon the application of two tones, with one tone having 1 V amplitude and $\delta = 4$ kHz, while the other tone (2 V) is swept over a $\Omega = 10$ kHz interval. The system exhibits a frequency doubling bifurcation route to chaos once the high amplitude tone is within its respective bistability region. This effect holds for a bidirectional frequency sweep, as shown in Fig. 1(e). The spectral response as well as the phase-space plots corresponding to a rotating frame periodic motion, period-doubling motion, and chaotic motion are shown in Fig. 1(f)-(h), and Fig. 1(i)-(k), respectively.

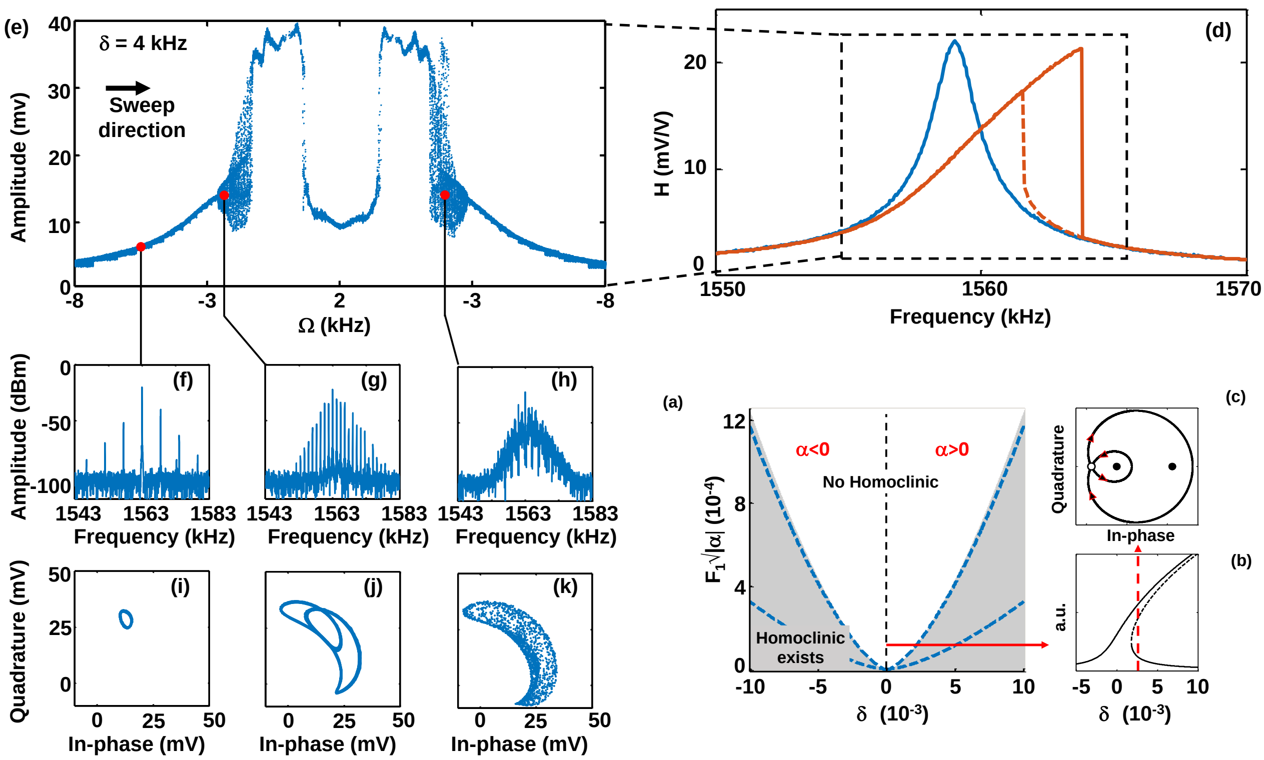

Figure 1: (a) Bistability map plotted as a function of non-dimensional force and detuning, showing the region of bistability for a lossless driven Duffing resonator (grey area), and for a low-loss (Q=1000) Duffing resonator (area within the dashed blue line). (b) Amplitude versus detuning response of a lossless Duffing taken for $F_1\sqrt{\alpha} = 1$. The corresponding phase-space plots for a detuning of $\delta = 2.5 \times 10^{-3}$ is shown in (c). The stable fixed points and the saddle point are shown as black and white dots respectively, as well as the homoclinic orbit. (d) Experimentally obtained frequency response sweep showing the linear regime (blue trace) and the Duffing regime (red trace). (e) Amplitude of the rotating-frame oscillations under the effect of a two-tone excitation, with one fixed tone ($\delta = 4$ kHz), and Θ swept between $[-8,\ 2]$ kHz, the frequency is swept in both directions with the forward and backward sweeps plotted side by side. The oscillations show Period 1, Period 2, and chaos, the spectral density and phase space plots of which examples are shown in (f)-(h), and (i)-(k), respectively.

Conclusions

In conclusion, this work presents an approach that leverages our understanding of the rotating frame approximation in order to generate chaos in nonlinear M/NEMS resonators using low power actuation. The chaos generation area within the parameter space is constrained using arguments from Melnikov's method, and the technique presented here is independent of the exact device design and scale, and therefore is generally applicable to low-loss nonlinear M/NEMS resonators.

References

[1] Bienstman J., Vandewalle J., Puers R. (1999) The autonomous impact resonator: a new operating principle for a silicon resonant strain gauge. *SENSOR ACTUAT A-PHYS* **66**:40-49.

[2] DeMartini B. E., Butterfield H. E., Moehlis J., Turner K. L. (2007) Chaos for a microelectromechanical oscillator governed by the nonlinear Mathieu equation. *J MICROELECTROMECH S*, **16**:1314-1323.

[3] Holmes P. (1979) A nonlinear oscillator with a strange attractor. *PHILOS T R SOC A*, **292**:419-448.