Controlling canard cycles

Hildeberto Jardon-Kojakhmetov*, and Christian Kuehn *
*Zentrum Mathematik, Technische Universität München, Boltzmannstr. 3, 85748 Garching bei München, Germany

Summary. Canard cycles are periodic orbits that appear as special solutions of fast-slow systems (or singularly perturbed Ordinary Differential Equations). It is well known that canard cycles are difficult to detect, hard to reproduce numerically, and that they are sensible to exponentially small changes in parameters. By combining Geometric Singular Perturbation Theory and state feedback control techniques, we design controllers that stabilize canard cycles of planar fast-slow systems with a folded critical manifold. As an application, we propose a controller that produces stable mixed-mode oscillations in the van der Pol oscillator.

Introduction

Fast-slow systems (also known as singularly perturbed ordinary differential equations are often used to model phenomena occurring in two or more time scales. Examples of these are vast and range from oscillatory patters in biochemistry and neuroscience all the way to stability analysis and control of power networks, among many others. The overall idea behind the analysis of fast-slow systems is to separate the behavior that occurs at each time scale, understand such behavior, and then try to elucidate the corresponding dynamics of the full system. Many approaches have been developed, such as asymptotic methods, numeric and computational tools and geometric techniques, see e.g. [4, 10].
Although the time scale separation approach has been very fruitful, there are some cases in which it does not suffice to completely describe the dynamics of a fast-slow system. The reason is that, for some systems, the fast and the slow dynamics are interrelated in such a way that some complex behavior is only discovered when they are not fully separated. An example of the aforementioned situation are the so-called *canards* [1]. Canards are orbits that, counter-intuitively, stay close for a considerable amount of time to a repelling set of equilibrium points of the fast dynamics. Canards are extremely important in the theory of fast-slow systems, and through them one can explain, for example, the very fast onset of large amplitude oscillations due to small changes of a parameter in neuronal models[3, 9], and the delayed loss of stability due to a slow passage through a singularity [11, 12]. However, due to their very nature, canard orbits are not robust, meaning that small perturbations may drastically change the shape of the orbit.
On the other hand the application of singular perturbation techniques in control theory is far-reaching. Perhaps, as already introduced above, one of the biggest appeals of the theory of fast-slow systems is the time scale separation, which allows the reduction of large systems into lower dimensional ones for which the control design is simpler. Applications range from the control of robots, all the way to industrial biochemical processes, and large power networks [7]. However, as already mentioned, not all fast-slow systems can be analyzed by the convenient time scale separation strategy, and although some efforts from very diverse perspectives have been made, a general theory that includes not only the regulation problem but also the path following and trajectory planning problems is, to date, lacking.
Here we merge techniques of fast-slow dynamical systems with control theory methods to develop controllers that stabilize canard orbits. The idea of controlling canards has already been explored in [2], where an integral feedback controller is designed for the FitzHugh-Nagumo model to steer it towards the so-called "canard regime". In contrast, here we take a more general and geometric approach by considering the canard normal form.

Main results

The formal statements of these results and the corresponding proofs are available in [5].

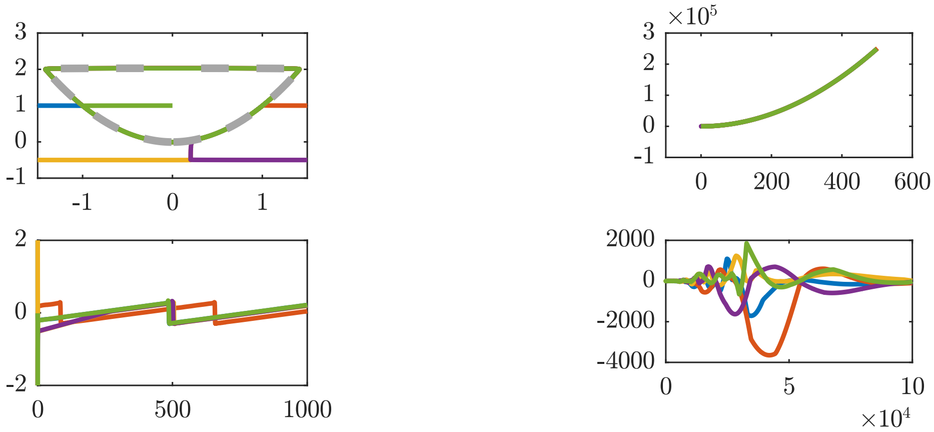
Consider the fast-slow control problem

$$\begin{aligned} x' &= -y + x^2 + u(x, y, \varepsilon, \alpha) \\ y' &= \varepsilon(x - \alpha), \end{aligned} \tag{1}$$

where $(x, y) \in \mathbb{R}^2$ are the fast and slow variables respectively, $\alpha \in \mathbb{R}$ is a parameter, $0 < \varepsilon \ll 1$ is the perturbation aprameter responsible for the time scale separation, and $u \in \mathbb{R}$ denotes a state-feedback controller.
Our main result is a controller u that renders canard cycles of (1) asymptitically stable as shown in Figure 1.
As an application we consider controlled the van der Pol equation

$$\begin{aligned} x' &= -y + x^2 - \frac{1}{3}x^3 + u \\ y' &= \varepsilon x. \end{aligned} \tag{2}$$

and design a controller that is able to produce in a robust way any type of Mixed-Mode Oscillation (MMO) allowed by the geometry of the critical manifold. An example of such MMO is shown in Figure 2.
Our main techniques of analysis are the blow-up method [6], canard theory [1, 8] and state feedback control design based on Lyapunov stability [13].



(a) (b)

Figure 1: In both columns we show, in the first row the (x, y) phase portrait of the closed-loop system (1) and in the second row the time-series of the corresponding controller. In all these simulations $\varepsilon = 0.01$. (a) The case of bounded canard cycles, where the desired canard cycle to be followed is shown in dashed-grey. (b) The maximal canard case, which is unbounded, and yet can be followed with a bounded controller.

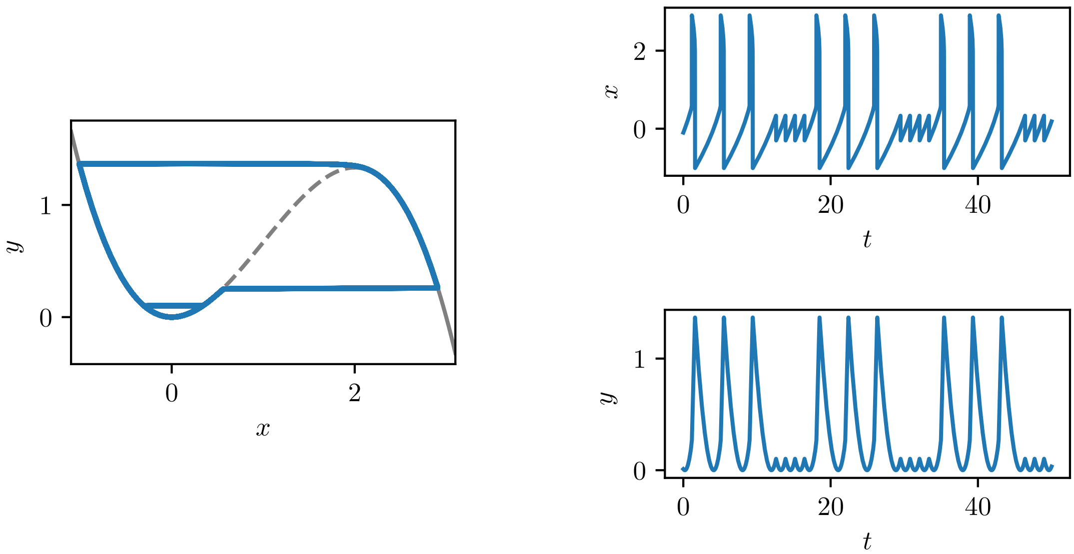


Figure 2: A sample of a Mixed-Mode Oscillation (MMO) with 3 Large Amplitude Oscillations (LAOs) and 4 Small Amplitude Oscillations (SAOs) produced by our controller. Such a controller has as parameters the number of LAOs, the number of SAOs, and the height of the canards.

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