On reduced-order models for resonant nonlinear dynamics: Galerkin truncation, nonlinear normal mode, and dominant spectrum decomposition

Tieding Guo[*], Giuseppe Rega[**]
[*]*College of Civil Engineering, Hunan University, Changsha, China*
[**] *Dipartimento di Ingegneria Strutturale e Geotecnica, Sapienza University of Rome, Rome, Italy*

Summary. It is well-known that perturbation analysis of reduced-order models (ROMs) of nonlinear structures produced by classical Galerkin truncation (using single or finite linear modes) might lead to erroneous results. At least three different approaches were proposed for resolving this issue in the literature, besides increasing retained modes (thus obtaining not a minimal ROM), namely, perturbation by directly attacking the continuous partial differential equations (PDEs), rectified Galerkin truncation, and nonlinear normal modes. The latter two give their ROMs but the first does not, although all the three lead to notably improved nonlinear responses. The three should be equivalent to each other in the sense of improved characterization of structure's nonlinearity, although at first glance they are quite different in their formulations. Our key observation is that the underlying essential similarity of the three resolutions is that the structure's dominant nonlinearity effects are always well captured *before* mode truncations (in distinct and subtle manners). Inspired by this similarity, we propose a new reduced-order modelling approach based upon dominant spectrum decomposition idea, with also comparisons of the above three existing methods being discussed. Explicitly, the key dominant dynamic patterns/features (indicated by their associated spectrum) inherent with nonlinear structures are captured before mode truncation. These dynamic patterns include not only the directly excited structural modes (always retained in classical Galerkin truncation), but also those dominant passive patterns which are slaved to the quadratic nonlinearity, sub-(super) harmonic excitations, or hard non-zero boundary conditions.

Basic formulation

Three typical one-dimensional scenarios in weakly nonlinear dynamics are discussed [1, 2], explicitly
(a) Hard sub-(super) harmonic excitation problem denoted by

$$\ddot{w}(x,t)+L[w]=F(x)\cos\Omega t+\varepsilon N_3[w]-2\varepsilon\mu\dot{w} \tag{1}$$

with boundary conditions $w(0,t)=w(1,t)=0$. Here $L[\cdot]$, $N_3[\cdot]$ are the structure's linear and geometric cubic-nonlinear operators, $F\sim O(1)$, $\Omega=3\omega_m+\varepsilon\sigma$ are the amplitude and frequency of a hard sub-harmonic external excitation, with ε being a small parameter for proper perturbation analysis, and σ a detuning parameter.

(b) Quadratic (and cubic) nonlinearity problem governed by

$$\ddot{w}(x,t)+L[w]=N_2[w]+N_3[w]+F\cos\Omega t-2\mu\dot{w} \tag{2}$$

with $w(0,t)=w(1,t)=0$. Here $w(x,t)\sim O(\varepsilon)$ is the structure's displacement, and $L[\bullet]$, $N_2[\bullet]$, and $N_3[\bullet]$ are the linear, quadratic and cubic (spatial) operators, respectively. $F\sim O(\varepsilon^3)$ is the excitation, with $\Omega=\omega_m+\varepsilon^2\sigma$.

(c) Hard sub-(super) harmonic moving boundary problem represented by

$$\ddot{w}(x,t)+L[w]=\varepsilon N_3(w,t)-2\varepsilon\mu\dot{w},\quad w(0,t)=0,\ w(1,t)=s(t)=S_0\cos\Omega t \tag{3}$$

where the boundary motion $s(t)=S_0\cos\Omega t$ is a hard sub-harmonic kinematic excitation with $S_0\sim O(1)$, $\Omega=3\omega_m+\varepsilon\sigma$. Here S_0, Ω are amplitude and frequency of boundary motion, respectively.

For the three problems above we point out that, essentially, one single structural mode, i.e., the *m*-th mode $(\omega_m,\ \phi_m)$, is directly excited and will thus survive in the corresponding ROMs in the absence of internal resonance, meaning that it is possible to use a single-mode Galerkin truncation like $w(x,t)\approx\phi_m q_m(t)$ for Eqs. (1) and (2) and $w(x,t)\approx\phi_m q_m(t)+\psi_0(x)s(t)$ for Eq.(3), where a shape function $\psi_0(x)$ is introduced for satisfying the non-zero boundary motion $s(t)$, with $\psi_0(0)=0,\ \psi_0(1)=1$. However, it turns out that the induced perturbation results do not agree with the direct perturbation outcomes (regarded as the most accurate), indicating that the single-mode truncation is incorrect and that more structural modes should be retained in the Galerkin truncation.

Error source analysis

Our observation is that, although only the *m*-th structural mode is directly excited, there are possibly other passive dominant dynamic patterns which should be captured. For example, if only the *m*-th mode is retained in the low-order Galerkin-reduced model, all the following response components, i.e., q_j, $j\neq m$, will be completely neglected [1, 2]

$$q_1=B_1e^{\mathrm{i}\omega_1T_0}+\hat{c}_1e^{\mathrm{i}\Omega T_0}+cc.,\quad q_2=B_2e^{\mathrm{i}\omega_2T_0}+\hat{c}_2e^{\mathrm{i}\Omega T_0}+cc.,\cdots q_j=B_je^{\mathrm{i}\omega_jT_0}+\hat{c}_je^{\mathrm{i}\Omega T_0}+cc.,\cdots \tag{4}$$

The key subtle point is that, although the free structural modal amplitudes $B_j\to 0,\ j\neq m$ will vanish eventually (not being directly or indirectly excited), certain forced components $p_j=\hat{c}_je^{\mathrm{i}\Omega T_0}+cc.,\ j\neq m$ might be non-trivial (say, be of comparable amount with respect to the retained components q_m and p_m). These non-trivial forced components $p_j = \hat{c}_j e^{\mathrm{i}\Omega T_0} + cc.$ are exactly what we meant by dominant passive dynamic patterns besides the retained (ω_m, ϕ_m), which could be caused by hard secondary excitations, quadratic nonlinearity and hard boundary motion. Explicitly,

$$F\cos\Omega t \text{ for Eq.(1)}, \quad N_2[\phi_m,\phi_m]q_m^2 \text{ for Eq.(2)}, \quad -\left(-\Omega^2 I + L\right)\psi_0(x) S_0 \cos\Omega t \text{ for Eq.(3)} \tag{5}$$

are the (non-secular) sources inducing non-trivial passive dynamic patterns, and should be fully captured. If single-mode truncation is used, only the *m*-th projected component of the terms in Eq.(5) will be considered. This is the error source ensuing from the perturbation analysis when using single-mode based ROMs.

Dominant spectrum decomposition and minimal ROMs [2]

We propose the following single-mode truncation corrected by the dominant spectral decomposition, i.e.,

$$w(x,t) = \phi_m(x)\cdot q_m(t) + \sum \Phi_{\Omega_i}(x)\cdot p_{\Omega_i}(t) \tag{6}$$

where $\left(\phi_m,\ q_m \sim e^{\mathrm{i}\omega_m t}\right)$ is the directly excited structural mode, and $\left(\Phi_{\Omega_i},\ p_{\Omega_i} \sim e^{\mathrm{i}\Omega_i t}\right)$ are the *i*-th forced components or passive dynamic patterns due to various sources denoted by Eq.(5) above, with $\{\Omega_1, \Omega_2, \cdots\}$ being the set of frequencies of these dynamic patterns, entering explicitly the low-order equation of motion as

$$\left(\partial^2/\partial t^2 + L\right)\sum \Phi_{\Omega_i}(x)\cdot p_{\Omega_i}(t) = \text{source terms in Eq.(5)} \tag{7}$$

Therefore, using Eqs. (6) and (7), we derive the new ROMs of the nonlinear structure denoted by

$$\left(\partial^2/\partial t^2 + L\right)\phi_m q_m(t) = \underbrace{0}_{\text{source terms in Eq.(5) eliminated}} + \varepsilon \mathrm{RT}\left[\phi_m, q_m, \Phi_{\Omega_i}, p_{\Omega_i}\right] + \mathrm{NST} \tag{8}$$

Perturbation analysis using ROMs in Eq.(8) agrees with the direct perturbations, as illustrated in Fig.1 below for the moving boundary problem in Eq.(3), where κ_{S2} is one key parameter of the modulation equations. Here RT and NST are short for resonant and non-secular terms, respectively. Note that discrete-1 uses $w(x,t) \approx \phi_m q_m(t) + \psi_0(x)s(t)$, with partially projected passive dynamic pattern captured, while discrete-2 uses $w(x,t) = \phi_m q_m(t) + \psi_0 s(t) + \Phi_\Omega(x) p_\Omega(t)$ with passive dynamic patterns fully captured, where $s(t) = p_\Omega(t)$, $\psi_0 + \Phi_\Omega = \Lambda_{B1}\left[g_\Omega(x,\xi)\right]$ and g_Ω is the steady Green's function [1,2]. Results will also be discussed by comparing with those based upon rectified Galerkin method [3] and nonlinear normal modes (NNMs) [4].

Note that the dominant spectrum decomposition technique above can be essentially used for reduced-order modelling of more general (strong/weak) nonlinear systems, if passive dynamic patterns can be explicitly obtained (numerical simulations might be employed for detections), although in this presentation we focus on approximate analytical passive patterns which can be derived in a perturbation formulation.

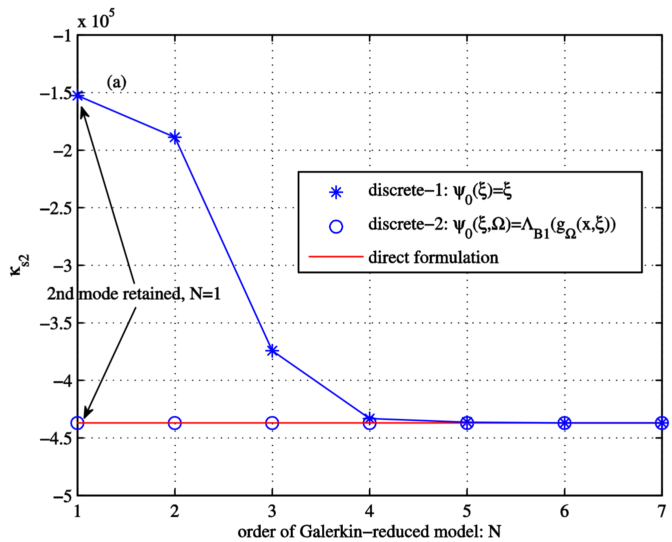

Fig.1 Convergence of modulation parameters in the modulation equations for the moving boundary problem [1]

References

[1] T.D Guo, G. Rega (2019), On direct and discretized perturbation formulations revisited for nonlinear structures' moving boundary problem, European Journal of Mechanics-A/Solids, submitted

[2] T.D Guo, G. Rega, H.J Kang (2019), Perturbation analysis using structure's Galerkin-truncated model: error source, physics interpretation, and a general correction procedure, Nonlinear Dynamics, submitted

[3] A. H Nayfeh (1998), Reduced-order models of weakly nonlinear spatially continuous systems, Nonlinear Dynamics, 15:105-125

[4] C. Touzé, M. Amabili, M O. Thomas (2008), Reduced-order models for large-amplitude vibrations of shells including in-plane inertia, Computer Methods in Applied Mechanics and Engineering, 197: 2030-2045