Controlled motion of two interacting particles on a rough inclined plane

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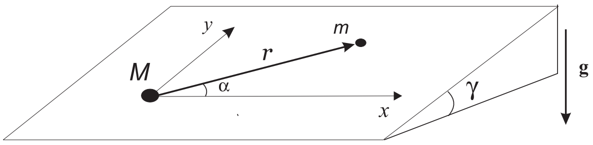
Summary. Two interacting particles on a rough inclined plane are considered. Coulomb's friction acts between the particles and the underlying surface. The system is controlled by the force of interaction of the particles. It is assumed that the parameters of the system are such that one of the bodies can be moved upward along a line of maximum slope provided that the other body is resting. The controllability of the system between two arbitrary states of rest is investigated. The system is proved to be controllable if the particles do not lie on a common line of maximum slope at the initial instant. A control algorithm that alternates quasistatic and fast modes of motion is constructed.

Statement of the problem

Consider a system of two particles on an inclined plane Π (Fig. 1a). Let m and M denote the masses of the particles ($m < M$), k the coefficient of Coulomb's friction between the particles and the underlying plane, γ the inclination angle of the plane, g acceleration due to gravity, $\mathbf{F}$ the interaction force applied by particle M to particle m. We assume that for $\mathbf{F} = 0$, both particles can stay at rest and that particle m can be moved from the state of rest by the force $\mathbf{F}$ upward along the line of maximum slope, while particle M does not move:

$$kM\cos\gamma \geq (M+m)\sin\gamma + km\cos\gamma. \tag{1}$$

Let the system under consideration be at rest at the initial instant. The aim of our study is to find out whether the system can be driven from the initial state to any other state of rest on the plane. For the horizontal plane ($\gamma = 0$), this is impossible. We are interested in the controllability of the system in principle. For this reason, we do not impose any constrains on the magnitude of the control force, allow instantaneous change in the positions of the particles, and assume that the particles may move through one another. If the particles at the initial instant rest on the common line of maximum slope, they cannot quit this line, and this case will not be considered. We will show that the system can be driven between



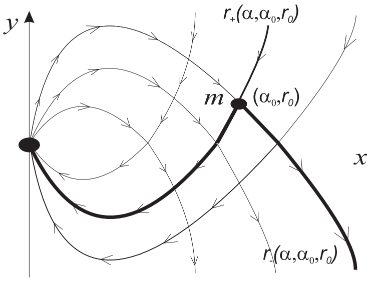


Figure 1: a) Two-particle system on an inclined plane, b) Quasistaic trajectories of particle m

the initial and terminal states by combining two types of motions: quasistatic motions and fast motions. The quasistatic motion is a slow motion that can be regarded as a continuous sequence of equilibria, while for fast motion we admit an instantaneous change in the positions of the particles.

Quasistatic motions

Inequality (1) implies that in quasistatic motions, only particle m moves, while particle M is at rest. Denote by L_M the line of maximum slope passing through the point M. Introduce in plane Π the coordinate frame Mxy (Fig. 1a). The axis y lies on line L_M and is directed upward. The trajectories of the quasistatic motion of particle m are defined by the equation

$$\frac{dr}{d\alpha} = \pm\frac{r\sqrt{1-a^2\cos^2\alpha}}{a\cos\alpha}, \quad a = \frac{\tan\gamma}{k}, \tag{2}$$

where r and α are the polar coordinates of particle m in plane Π related to the pole M and the polar axis Mx. When moving quasistatically, particle m cannot intersect line L_M; therefore we assume $\alpha \in (-\pi/2,\, \pi/2)$. Sign minus on the right-hand side in Eq. (2) corresponds to the repulsive motion when the interaction force $\mathbf{F}$ applied to particle m acts from M toward m, while sign plus corresponds to the attractive motion. Equation (2) has a closed-form solution in terms of elementary functions. Denote by $r_\pm(\alpha, \alpha_0, r_0)$ the solution of Eq.(2) subject to the initial conditions $r(\alpha_0) = r_0$. The function r_+ (r_-) monotonically increases (decreases) as α increases in the interval $(-\pi/2, \pi/2)$. The functions r_+ and r_- have the following properties:

$$\lim_{\alpha\to\pi/2} r_-(\alpha) = 0, \quad \lim_{\alpha\to-\pi/2} r_-(\alpha) = \infty, \quad \lim_{\alpha\to-\pi/2} r_-(\alpha)\cos\alpha = \infty. \tag{3}$$

$$\lim_{\alpha\to\pi/2} r_+(\alpha) = \infty, \qquad \lim_{\alpha\to-\pi/2} r_+(\alpha) = 0, \qquad \lim_{\alpha\to\pi/2} r_+(\alpha)\cos\alpha = \infty. \tag{4}$$

$$r_-(\alpha,\alpha_0,r_0) = r_+(-\alpha,-\alpha_0,r_0). \tag{5}$$

According to the properties of Eqs. (3)–(5), the trajectories of the quasistatic motion of particle m on plane Π have the shape shown in Fig. 1b. For each point $(\alpha_0,\, r_0)$ on the plane, one can indicate an area (attainable area) to each point of which particle m can be driven quasistatically. This area is bounded by the curves $r_\pm(\alpha,\alpha_0,r_0)$ (thick lines in Fig. 1b). One can drive particle m to any internal point (α_1,r_1) of the attainable area using one switching between the repulsive and attractive motions; the particle moves first along the curve $r_-(\alpha,\alpha_0,r_0)$ or $r_+(\alpha,\alpha_0,r_0)$ and then, respectively, along the curve $r_+(\alpha,\alpha_1,r_1)$ or $r_-(\alpha,\alpha_1,r_1)$. In particular, using one switching, one can get quasistatically to the point $(\alpha_0-\delta\alpha,\, r_0)$, $\delta\alpha \ll 1$, from the point $(\alpha_0,\, r_0)$. Then, similarly, using one switching, one can get to the point $(\alpha_0-2\delta\alpha,\, r_0)$ from the point $(\alpha_0-\delta\alpha,\, r_0)$, and so on. By letting $\delta\alpha\to 0$, we obtain that the trajectory of particle m can be made arbitrarily close to the circular arc of radius r_0; the angle α monotonically decreases, approaching but not reaching a value of $-\pi/2$. Therefore, particle m can be driven from the point $(\alpha_0,\, r_0)$ quasistatically along a trajectory arbitrarily close to the circular arc $r=r_0$, $\alpha\in(-\pi/2,\alpha_0]$, with monotonically decreasing angle α. We will call such a motion the quasistatic motion along a circumference.

All the aforesaid remains valid for the quasistatic motion for $\alpha\in(\pi/2,\,3\pi/2)$. In this case, the repulsive and attractive trajectories will be symmetric about the My-axis to the respective trajectories for $\alpha\in(-\pi/2,\,\pi/2)$; particle m can be driven along a circumference, with α monotonically approaching but not reaching a value of $3\pi/2$.

Fast motions. An algorithm for driving the system to the terminal state

By fast motions we understand the motions that drive the system between different states of rest in an infinitesimal time. The force of interaction between the particles for such motions is much larger than the external friction forces; therefore, the system's center of mass and the line that connects the particles are fixed. We allow the particles in the fast motion to pass through one another, changing as a result the direction of the vector $\overrightarrow{Mm}$ to the opposite one. By means of the fast motion, we can move particle M to any position on the initial line Mm.

By alternating fast and quasistatic motions one can move particle m to any position on the plane, with particle M remaining arbitrarily close to its initial position. We will show this for the case where the initial and terminal positions of particle m belong to different half-planes with respect to line L_M. We assume for definiteness that $\alpha\in(\pi/2,3\pi/2)$ for the initial positions and $\alpha\in(-\pi/2,\pi/2)$ for the terminal position. We will show first that particle m can be brought onto a semicircumference of an arbitrarily small radius on the right half-plane, i.e., to any position $(\alpha^*,\, r^*)$ such that $r^*=\varepsilon$, $\varepsilon\ll 1$, and $\alpha^*\in(-\pi/2,\pi/2)$, while the change in the position of particle M is small. To this end we at the first stage move quasistatically particle m toward particle M until the distance r between the particles becomes $r=\varepsilon$. If at this instant the angle α does not satisfy the inequality $|\alpha-3\pi/2|\le|\pi/2-\alpha^*|$, we move particle m along a circumference until this inequality holds. After this, we perform the fast motion as a result of which particles m and M change their positions on the line Mm to the positions that are symmetric about the center of mass. The change in the position of particle M at this stage is small (the distance moved by this particle is less than ε), while the distance between the particles does not change. At the final stage, particle m moves quasistatically clockwise (with the angle α monotonically decreasing) until the angle α becomes equal to the desired value α^*. Thus we proved the possibility for particle m to be driven to an arbitrary point of a circumference of small radius on the right half-plane. Taking into account the fact that the quasistatic repulsive and attractive trajectories that go out from all points $(\alpha^*,\, r^*)$, $\alpha^*\in(-\pi/2,\pi/2)$, $r^*=\varepsilon$, sweep the entire right half-plane, we conclude that particle m can be driven quasistatically into any position on the right half-plane. Somewhat simplifying, we can regard the algorithm presented above as driving particle m onto particle M followed by the motion of particle m along an arbitrary trajectory of quasiatatic repulsion that goes out from the origin.

In conclusion, we describe an algorithm that drives the system to the desired terminal state. We will confine ourselves to the case where the terminal positions of the particles do not lie on the common line of maximum slope. At the first stage, by alternating fast and quasistatic motions as was described above, we bring the system to a position in which the line that connects the particles passes through the terminal position of particle M. Then by means of fast motion we move particle M to its terminal position. Finally, by using the algorithm described above, we move particle m to the desired terminal state; at this stage, the change in the position of particle M is arbitrarily small.

Conclusions

It is proved that if at the initial instant the particles do not lie on the common line of maximum slope, the system can be driven into an arbitrarily small neighborhood of any terminal position on an inclined rough plane by combining quasiastatic and fast motions. A system of two interacting particles is a simple model of limbless worm-like crawlers. This biomimetic principle of motion can be used for mobile microrobots. It is important that when on a horizontal plane, a two-particle locomotion system that is in a state of rest at the initial time instant can move only along a line that connects the initial positions of the particles, whereas on an inclined plane, the system can be driven to any terminal position.