Dynamical analysis of TET in a non-smooth vibro-impact system

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Summary. Applying a novel analysis we study targeted energy transfer (TET) in a non-smooth vibro-impact system (VI), comprising a ball freely moving within a frictionless slot made within a harmonically excited large mass. Both the ball and the mass can move independently between inelastic collisions. The proposed semi-analytical approach allows analyzing TET in the discontinuous piece-wise linear systems exactly, in contrast with previous studies via approximations related to a specific state. We obtain parameter ranges for the impact pair to effectively transfer energy to it from the base excited system via a series of impacts.

Problem statement

Targeted energy transfer may be viewed as an extension of a classical linear tuned mass damper (TMD) theory, where a second mass-spring system is added to an original single-degree-of-freedom (SDOF) oscillator to avoid a high-amplitude response near its resonance frequency [1, 2]. The parameters of the resulting two DOF (TDOF) system are selected so achieve a low energy state in the original system, while the added mass-spring subsystem stores the energy of the entire TDOF system. While TMD is a standard approach for vibrational mitigation and energy transfer, used by engineers for over 100 years, it has no direct analogy for nonlinear systems. Without an exact analytical solution for the classical non-linear TDOF system there is no “obvious” way to choose the systems’ parameters for transferring energy from a forced oscillator to a nonlinear energy sink via nonlinear coupling, see [3, 4] and references therein. Analyses are generally based on weakly nonlinear approximations or model reductions. Parametric studies are then limited requiring simulations for explorations of the full nonlinear behavior.

Alternatively, we note that energy transfer in continuous nonlinear systems is a special case of more general nonlinear systems, which can be non-smooth and discontinuous. Considering TET via persistent vibro-impact motion (e.g. as in [5, 6]) we focus on the non-smooth impact pair as in [5, 7], with a small mass ball traveling freely within a slot of length $2b$ made in the large excited mass. The dynamics of the displacements q_1, q_2 of the large and small masses with mass M and m respectively, complemented by the impact conditions at $|q_1 - q_2| = b$, is described by the following equations:

$$M\ddot{q}_1 + c\dot{q}_1 + kq_1 = kE\sin(\omega t + \phi) + c\omega E\cos(\omega t + \phi) \qquad \ddot{q}_2 = 0 \tag{1}$$

$$(\dot{q}_1^+ - \dot{q}_2^+) = -r(\dot{q}_1^- - \dot{q}_2^-) \qquad M\dot{q}_1^+ + m\dot{q}_2^+ = M\dot{q}_1^- + m\dot{q}_2^-, \qquad {}^{-(+)}\text{ before (after) impact} \tag{2}$$

where k is the elastic spring coefficient, E, ω, ϕ are the excitation amplitude, frequency, and initial phase, r is the restitution coefficient, t is the time, the dot indicates derivatives with respect to time. The instantaneous impact conditions (2) capture the two-way energy exchange between the masses, in contrast to the extensive literature on bouncing ball dynamics, where the mass of the ball is typically assumed as negligible [8].

Analytical approach

Within a non-dimensionalized framework, with parameters $\mu = \frac{m}{M}$, $A = \frac{E}{b}$, $\omega_0 = \sqrt{\frac{k}{M}}$, $\Omega = \frac{\omega}{\omega_0}$, $\lambda = \frac{c}{M\omega_0}$, state variables $x_1 = \frac{q_1}{b}$, $x_2 = \frac{q_2}{b}$ and rescaled time $\hat{t} = \omega_0 t$, and dropping ˆ’s for the remainder, we integrate the equations of motion (1), and apply the impact conditions (2) when the ball impacts either end of the cavity. This yields the map of the state from the previous impact time t_{k-1} to the next impact time t_k in terms of all system parameters

$$\begin{aligned}
x_{1,k} &= a_1 e^{\frac{-\lambda t_k}{2}}\sin(\gamma t_k) + a_2 e^{\frac{-\lambda t_k}{2}}\cos(\gamma t_k) + b_1\sin(\Omega t_k + \phi) + b_2\cos(\Omega t_k + \phi) \\
\dot{x}_{1,k} &= a_1 e^{\frac{-\lambda t_k}{2}}\left(-\frac{\lambda}{2}\sin(\gamma t_k) + \gamma\cos(\gamma t_k)\right) + a_2 e^{\frac{-\lambda t_k}{2}}\left(-\frac{\lambda}{2}\cos(\gamma t_k) - \gamma\sin(\gamma t_k)\right) + b_1\Omega\cos(\Omega t_k + \phi) \\
&\quad - b_2\Omega\sin(\Omega t_k + \phi), \\
x_{2,k} &= x_{2,k-1} + \left(\frac{1+r}{1+\mu}\dot{x}_{1,k-1} + \frac{\mu - r}{1+\mu}\dot{x}_{2,k-1}\right)\cdot(t_k - t_{k-1}) \\
\dot{x}_{2,k} &= \frac{1+r}{1+\mu}\dot{x}_{1,k-1} + \frac{\mu - r}{1+\mu}\dot{x}_{2,k-1}
\end{aligned} \tag{3}$$

Here the conditions are in terms of $x_{j,k}, \dot{x}_{j,k}$ the displacements and velocities at the impact time t_k, corresponding to the end of each sub-interval of continuous motion, so that the superscript $^-$ as in (2) can be dropped without loss of generality. The coefficients a_l,b_l are functions (not given here) of the previous time t_{k-1} and states $\dot{x}_{j,k-1}x_{j,k-1}$. Taking $|x_{1,k-1} - x_{2,k-1}| = 1$ and $|x_{1,k} - x_{2,k}| = -1$ yields the map P_1 for the motion from left to right in the capsule. Similarly

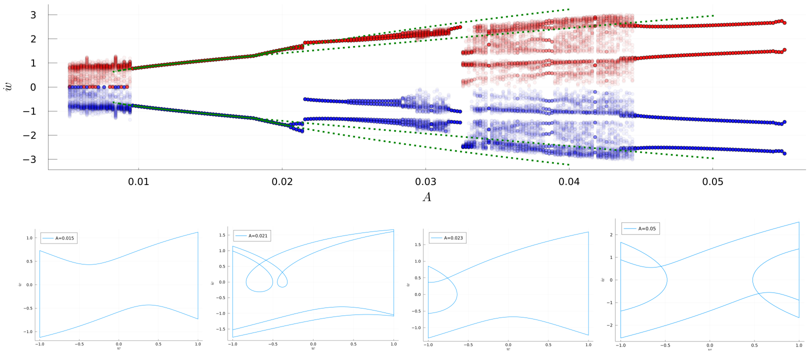

Figure 1: Upper: Bifurcation diagram of relative impact velocity $\dot{w}$ for (red (blue) is left (right) impact obtained numerically) vs. forcing amplitude A; Green solid (dotted) lines are stable (unstable) 1:1 solutions obtained analytically. Lower: Phase planes with $\dot{w}$ vs w at $A = .015$, $.021$, $.023$ and $.05$ showing 1:1, 1:1/2T, 2:1, and 2:2 periodic solutions, respectively.

we determine maps for the other motions, e.g. P_2 right to left, and P_3 (P_4) for successive impacts on left (right). Thus it is possible to construct a sequence of maps describing the different periodic motions, whether successive impacts occur on the same or different ends of the slot. We use the notation n:m/pT corresponding to pT-periodic solutions with n (m) impacts on the left (right) of the capsule, given an external forcing with period T on the large mass. Then 1:1 solutions, the simpler case where the ball has alternating impacts on either end of the capsule, are determined from the system (3) given by the composition $P_1 \circ P_2$ together with the periodicity condition. Similarly 2:1 solutions are obtained from the combined equations from the composition $P_3 \circ P_1 \circ P_2$ plus periodicity, and so on for other n:m behaviors. From these compositions plus periodic conditions we obtain the period solution in terms of state vectors $\mathbf{H}_k^* = (\phi_k, \dot{x}_{1,k}, \dot{x}_{2,k}, x_{1,k})$, where the phase shift ϕ_k of the impact is relative to the forcing. Based on these analytical results for the periodic solution, we can also study their linear stability [9], and thus identify analytically both traditional bifurcations such as period doubling and discontinuity-induced bifurcations. The procedure described in this section is also applied to obtain semi-analytical solutions of a soft impact model and are used to analyse the compliant system dynamics.

Comparison of Numerical and Analytical results

Figure 1 shows the numerically obtained bifurcation structure of the relative impact velocity $\dot{w} = \dot{x}_{1k} - \dot{x}_{2k}$ vs. forcing amplitude A, illustrating 1:1, 1:1/2T, 2:1, and 2:2 solutions, as well as chaotic behavior occurring for different A. Stable and unstable 1:1 solutions obtained analytically are shown, indicating good agreement between analytical and numerical results for the stable 1:1 periodic solutions, as well as the regions for instability. From the results we can calculate performance based on energy transfer efficiency, for example, kinetic energy transferred relative to work done by the excitation. This measure (not shown) indicates improved performance via 1:1 type periodic solutions, with some reduced performance at non-smooth bifurcations such as 1:1 to 2:1 transitions.

Conclusions

We study targeted energy transfer in a VI system by expressing the inter-interval dynamics as a sequence of maps. We capture physically-relevant motions via the derived semi-analytical approach that reveal critical parameter dependencies of the dynamics and energy transfer. Developing this approach within the instantaneous (hard) impact condition allows us to generalize to compliant impact conditions.

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