Dynamic non-smooth fold bifurcations influenced by oscillations and noise

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Summary. We contrast the behavior of dynamic bifurcations for smooth and non-smooth fold bifurcations in the presence of oscillatory and noisy forcing. Dynamic bifurcation refers to the state transition or "tipping" that takes place when a parameter slowly varies through a value corresponding to a bifurcation in the static model. Note that the dynamic models correspond to non-autonomous systems with multiple time scales. Through a canoncial model with an underlying static nonsmooth fold bifurcation, we see that the transition is delayed as in the smooth case, depending on the rate of change of the parameter, but the functional form for the non-smooth case is different. We also compare and contrast how oscillatory forcing can shift the tipping or dynamic bifurcation, in both smooth and NS cases. We extend to a higher degree of freedom models with NS fold bifurcations, studying a Stommel model, a two-box ocean model for temperature and salinity. Within this higher dimensional model, we find additional differences for the non-smooth case in the presence of certain types of random forcing.

Contrasts of tipping points in smooth and non-smooth fold bifurcations

We consider a non-autonomous single degree of freedom (DOF) model ,

$$\dot{x} = -\mu + 2|x| - x|x| + A\sin(\Omega t), \quad \dot{\mu} = -\,\epsilon, \qquad x(0) < 0, \quad \mu(0) > 0, \qquad \epsilon \ll 1, \tag{1}$$

with A and $\Omega = \epsilon^{-\lambda}$ the amplitude and frequency, respectively, of the oscillatory forcing and slowly varying bifurcation parameter μ. For the static system $\epsilon = 0$ there is an underlying non-smooth (NS) fold bifurcation at $\mu = x = 0$ as shown in Fig. 1. Using a series of multiple scale analyses, we obtain approximations for the dynamic bifurcations, often called

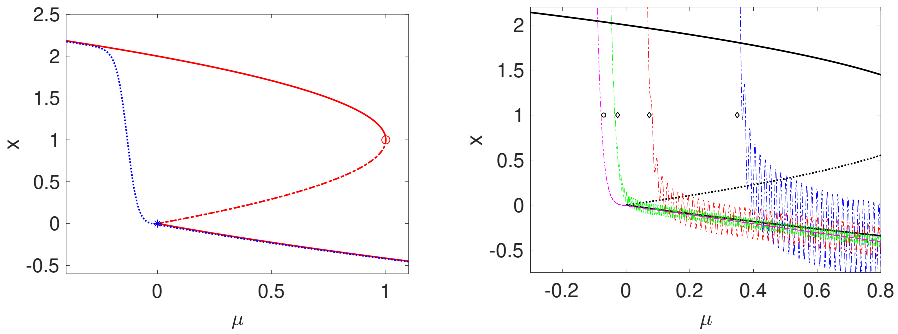

Figure 1: (LEFT) The bifurcation diagram for (1) showing upper and lower equilibrium branches (solid lines) and the unstable middle branch (dash-dotted line). The NS static fold bifurcation is at (0,0) (blue $*$);the smooth fold bifurcation is at (1,1) (red o). The numerical solution (blue dotted line) to (1) is for $A = 0$ and $\epsilon = .05$. (RIGHT). Simulations of (1), superimposed on the static bifurcation curve (black lines), Diamonds: analytical predictions μ_{tip} for the DB/TP for $\lambda \leq 1$ (simulation in red and green for $A = 2$ and $\lambda = .7, 1$, respectively, and in blue for $A = 5$ and $\lambda = .7$); circle o: analytical prediction of μ_{sv}, (simulation in magenta for $A = 2$, $\lambda = 2$)

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tipping points (DB/TP), with and without the oscillatory forcing. Here x follows the lower equilibrium branch $x < 0$ as μ decreases. Using the multiple scales expansion $x(t,T) = -\epsilon^{\lambda} A\cos T + \epsilon^{q_1} y_1(t,T) + \ldots \epsilon^{q_2} y_2(t,T) + \ldots$ for $1 \ll \Omega = \epsilon^{-\lambda}$ and $T = \Omega t$, we obtain the DB/TP

$$A \neq 0, \;\; 0 < \lambda \lesssim 1 \qquad \mu_{\text{tip}} \;\; = \;\; \left(\frac{\epsilon^2}{\Omega}\right)^{1/3} \left(\frac{\pi|A|}{2}\right)^{1/3} \xi_r + \frac{4|A|}{\pi\Omega} \qquad \text{Ai}(\xi_r) = 0 \text{ for } \xi_r = -2.33811\ldots \tag{2}$$

$$A = 0, \;\; 0 < \lambda \qquad \mu_{\text{sv}} \;\; \sim \;\; \frac{1}{2}\epsilon\log(\epsilon). \qquad \text{(Ai is the Airy function)} \tag{3}$$

Fig. 2 compares the analytical and numerical results for these DB/TP's. For small frequencies the NS DB/TP is advanced relative to the static $\mu = 0$, while for larger frequencies (larger λ or smaller A/Ω), the DB/TP is positive, with its asymptote at μ_{sv} corresponding to $A = 0$. Note that μ_{tip} depends on the ratio A/Ω. Figs. 1 and 2 illustrate advanced DB/TP for larger A/Ω and lagged DB/TP for smaller A/Ω. We contrast these results with the DB/TP for the canoncial smooth dynamic fold bifurcation studied in [1]-[2], $x'(t) = a(\epsilon t) - x^2$, $a = a_0 - \epsilon t$ for $\epsilon \ll 1$. For $A \neq 0$, in both smooth and NS cases, the DB/TP's are the sum of two contributions, one negative contribution which corresponds to a lag due to slowly varying μ, and a second positive contribution corresponding to an advance in tipping due to the oscillations. Thus we have a competition between influences generating advances and lags, with different parametric dependencies. Analogous to the NS case, the DS/TP value of a in the smooth case depends on A/Ω and ϵ, but with a different functional dependency. For example, the asymptote for small A/Ω in the smooth case is $0 > a_{sv} = O(\epsilon^{2/3})$, in contrast to $\mu_{sv} = O(\epsilon\log\epsilon)$. Thus the lag in the DB/TP is larger in the smooth case, as shown Fig. 2 LEFT.

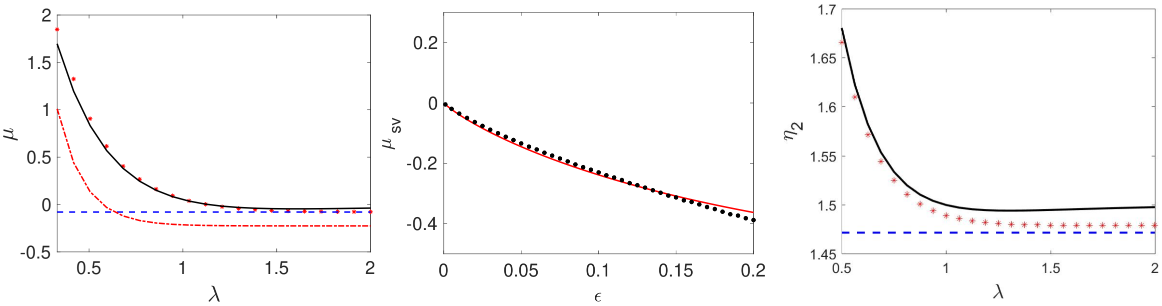

Figure 2: LEFT: For $A = 5$, $\epsilon = .03$. comparison of the critical value $\mu_{\rm tip}$ (black solid line) valid for $\lambda \lesssim 1$ and the limiting $\mu_{\rm sv}$ for larger λ (blue dotted line). Red stars indicate tipping in the numerical solution to (1), corresponding to the value of μ at which x reaches 1. The red dash-dotted line is the analogous results for the smooth fold bifurcation analyzed in [2], for comparison. CENTER: The tipping value for $\mu_{\rm sv}$ approximated by (3) (solid red line) and compared with the numerical result from (1) (black dots) with $A = 0$, taking $x_{\rm tip} = 1$. RIGHT: Same as LEFT, but for (4).

Applications in larger systems

We apply a similar approach in higher DOF models, illustrated via the non-dimensionalized Stommel box model for thermohaline ocean circulation [3],

$$\begin{aligned} &\dot{V} = \eta_1 - \eta_2 + \eta_3(T - V) - T - V|V| + A\sin(\Omega t), \qquad \dot{\eta_2} = -\epsilon \\ &\dot{T} = \eta_1 - T(1 + |V|) + B\sin(\Omega t), \qquad T(0) = T_i, \quad V(0) < 0, \quad \eta_2(0) = \eta_{2i} > \eta_1\eta_3\,. \end{aligned} \tag{4}$$

Here V is the difference between non-dimensional temperature T and salinity S, with underlying static bifurcation structure of V vs. η_2 similar to that of (1) in Fig. 1. Then the equilibrium branches for $V > 0$ ($V < 0$) corresponds to a temperature (salinity)-dominated state. There is slow variation in time for $\epsilon \ll 1$ of the bifurcation parameter η_2 and high frequency oscillations $\Omega \gg \epsilon^{-\lambda}$ for $\lambda > 0$ with amplitudes A and B. Model parameters η_1 and η_3 are positive constants. The varying parameter η_2 is related to the freshwater flux, as also studied in Roberts [4]. Variations in (η_1, η_3) are captured by nonzero A, B, analogous to observed behavior in Huybers [5]. Using the approaches developed for (1), we obtain approximations for the DB/TP $\eta_{2\rm tip}$ as shown in Fig. 2 RIGHT for the NS dynamic bifurcation, capturing the transition from solutions near the salinity-dominated branch for $V < 0$. Note, as in Fig. 2 LEFT, an advanced DB/TP relative to the static bifurcation point $\eta_{2c} \approx 1.5$ occurs for larger values of A/Ω. Lagged DB/TP occurs for smaller A/Ω, with an asymptote to the lag corresponding to slow variation of η_2 and $A = 0$.

Potential influence of coherence resonance in non-smooth dynamic bifurcations

For the model (4) forced by white noise rather than oscillatory forcing, there is the potential for a coherence-resonance (CR)-driven advance of the tipping point. A linear analysis of the salinity-dominated branch for $V < 0$, with static NS bifurcation at $\eta_2 = \eta_{2c}$, shows that the corresponding eigenvalues can be either real or complex. This behavior is in contrast to a smooth fold bifurcation with real eigenvalues of its near-by linearized system, e.g. as is the case for the temperature-dominated branch of (4) for $V > 0$. For the salinity-dominated branch of (4) for $V < 0$, the eigenvalues are typically complex as η_2 approaches η_{2c}, allowing for CR in which noise excites the frequency corresponding to the complex part of the eigenvalues, even if the real part is negative. Using a measure of CR based on the power spectral density of the fluctuations about the attracting salinity-dominated branch, we identify parameter ranges for which this CR produces a large probability of advanced tipping to the temperature-dominated state. (This is also joint work with Ziming Yin, now a co-op at Scotiabank in Toronto, CA.)

Conclusions

Via multiple scale analyses we capture the different parametric dependencies of dynamic bifurcations, i.e. tipping points, in forced systems with smooth and non-smooth (NS) dynamic fold bifurcations. For $O(1)$ forcing amplitude with high frequency oscillations, the advance in the tipping point is larger in the NS case, since the contributions from the lag due to slow variation of parameters is reduced relative to the smooth case. Furthermore, we find that coherence resonance-driven tipping can advance NS fold bifurcations since switching surfaces rather than loss of local attraction to the equilibrium state generates the bifurcation. Then the eigenvalues describing the local behavior near the NS fold bifurcation may be complex, so that the noise can excite frequencies corresponding to the imaginary part of these eigenvalues.

References

[1] R. Haberman, SIAM J. Appl.Math. **37** 1, 69–106 1979.

[2] J. Zhu, Jielin, R. Kuske, T. Erneux, SIAM J. Appl. Dyn. Sys. **14**, 4, 2030–2068, 2015.

[3] H. A. Dijkstra, Nonlinear climate dynamics, Cambridge University Press, 2013.

[4] A. Roberts and R. Saha, Climate Dynamics **48**, 7-8, 2123–2134, 2017,

[5] P. Huybers, Peter and C. Wunsch, Nature **434**, 7032, 491, 2005.