On solving one-dimensional wave equations subject to nonclassical and to nonlinear boundary conditions

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Summary. In this paper it will be shown how characteristic coordinates, or equivalently how the well-known formula of d'Alembert can be used to solve initial-boundary value problems for wave equations on semi-infinite intervals or on fixed, bounded intervals involving non-classical and nonlinear boundary conditions. It will be shown that solutions or approximations of solutions for wave-problems can be constructed for a much larger class of problems than generally is assumed. In this paper linear and nonlinear mass-spring-damper systems will be considered at the boundary, and it will be shown how (approximations of) solutions can be constructed.

Introduction and overview

The study of a one-dimensional wave equation such as

$$u_{tt} - u_{xx} = 0 \quad , \quad t > 0, -\infty < x < \infty , \tag{1}$$

goes back to the middle of the 18th century when d'Alembert solved an initial value problem for Eq. (1) on an infinite interval (that is, on $-\infty < x < \infty$) by using characteristic coordinates. In Eq. (1) $u = u(x,t)$ is a displacement (usually a lateral displacement of a string), *x* is a space coordinate, and *t* is time. When the initial displacement, and the initial velocity are given by

$$u(x,0) = f(x), \text{ and } u_t(x,0) = g(x) \ , \tag{2}$$

respectively, one obtains as solution the well-known formula of d'Alembert

$$u(x,t) = \tfrac{1}{2} f(x+t) + \tfrac{1}{2} f(x-t) + \tfrac{1}{2}\int_{x-t}^{x+t} g(s)\,ds. \tag{3}$$

On a semi-infinite interval (that is, for instance on $x > 0$) this formula (3) can also be used to solve an initial value problem for a wave equation. For a Dirichlet type of boundary condition at $x = 0$, or for a Neumann type of boundary condition at $x = 0$, it is also well-known that the functions in the classical formula of d'Alembert should be extended as odd, or as even functions in *x*, respectively. How the functions should be extended for a Robin type of boundary condition (with constant coefficients) at $x = 0$, is less well-known, but is was already discovered at the end of the 19th century by Bryan. Recently in [1] and [2] the extension procedures on semi-infinite intervals for problems with a linear mass-spring-damper boundary condition at $x = 0$, were presented for a string equation and for an axially moving string equation, respectively.

On a bounded interval (that is for instance on $0 < x < L < \infty$) the classical formula of d'Alembert can also be used to solve an initial value problem for a wave equation. In the literature only the cases where one has Dirichlet and/or Neumann boundary conditions, are solved by using the formula of d'Alembert, and leads to odd and/or even periodic extensions of the functions in the formula of d'Alembert. For other boundary conditions the formula of d'Alembert is not used, most likely, because it is not (well) known how to extend the functions in the formula of d'Alembert for other boundary conditions than those of Dirichlet type or of Neumann type.

Usually the method of separation of variables, or the (equivalent) Laplace transform method is used to solve initial value problem for a wave equation on a bounded interval for various types of boundary conditions with constant coefficients. However, when a Robin boundary condition with a time-dependent coefficient is involved in the problem, then the aforementioned methods are not applicable. In [3] it has been shown how characteristic coordinates or equivalently, how the classical formula of d'Alembert can be used to solve an initial value problem for a wave equation on a bounded, fixed interval with at one endpoint a Dirichlet type of boundary condition, and at the other end a Robin type of boundary condition with a time-dependent coefficient. The Robin boundary condition with a time-dependent coefficient is an interesting one to study from the applicational (and from the mathematical) point of view. When one considers the transversal vibrations of a string which at one end is attached to a spring for which the stiffness properties change in time (due to fatigue, temperature change, and so on), then a Robin type of boundary condition is obtained with a time-varying coefficient. But also in the study of longitudinal vibrations of axially moving strings with time-varying lengths (as simple models for vibrations of elevator or mining cables), one obtains, after some transformations as a first order approximation of the problem, a wave equation for which at one end a Robin type of boundary condition with a time-varying coefficient has to be satisfied.

For nonlinear or weakly nonlinear boundary conditions not so many results are known. The reader is referred to [4] (and the references in [4]) for some recent and historical approaches that have been used. In this paper a problem for a wave equation on a semi-infinite domain will be discussed. The boundary condition is assumed to be weakly nonlinear and it will be explained how a multiple time-scales perturbation method can be applied to construct approximations of the solution of the problem.

In this paper our recent results on the applicability of the formula of d'Alembert (as discussed above) will be presented. For details the reader is referred to our recent publications [1, 2, 3].

Four classes of problems

In this paper four classes of problems will be explained in detail. In the first class the vibrations of a semi-infinite string (that is, equation (1) with $x > 0$) will be studied with a spring-damper system attached at $x = 0$, that is, a boundary condition like

$$Tu_x(0,t) = ku(0,t) + \alpha\, \mathrm{u}_t(0,\mathrm{t}) \tag{4}$$

will be considered, where T is the tension in the string, k the stiffness of the spring, and α the damping coefficient of the dashpot. By using characteristic coordinates or equivalently, by using d'Alembert's formula (3) it will be shown that the extensions of the functions f and g in (3) for negative arguments in the functions f and g, have to satisfy nonhomogeneous, first order ordinary differential equations (ODEs). The solutions of these ODEs can readily be obtained, and so, reflected waves and energy dissipation can be determined.

In the second class of problems the vibrations of a semi-infinited string will be studied with a mass-spring-damper system attached at $x = 0$, that is, a boundary condition like

$$mu_{tt}(0,\mathrm{t}) = \mathrm{Tu}_x(0,t) - ku(0,t) - \alpha u_t(0,t) \tag{5}$$

will be considered, where m is the mass in the attached system at $x = 0$. In this case the extensions of the functions of f and g have to satisfy nonhomogeneous, second order ODEs with constant coefficients. Again reflected waves and damping properties can be determined.

In the third class of problems the vibrations of a finite string (that is, equation (1) with $0 < x < L < \infty$) will be studied with a fixed end at $x = 0$ and a spring (with a time-varying stiffness $k(t)$) attached at $x = L$, that is, with the boundary conditions

$$\begin{aligned} u(0,t) &= 0, \\ Tu_x(L,t) &= -k(t)u(L,t). \end{aligned} \tag{6}$$

It will be shown that the method of separation of variables can not be applied, but d'Alembert's formula (3) can be used. It will turn out that the extensions of the functions f and g (outside the interval $[0, L]$) have to satisfy nonhomogeneous first order ODEs and have to satisfy a certain iteration process. Again properties of the solutions can be obtained from d'Alembert's formula (3).

In the fourth class of problems the vibrations of a semi-infinite string will be studied for $x > 0$ with a (weakly) nonlinear mass-spring-damper system attached at $x = 0$, that is, a boundary condition like

$$mu_{tt}(0,t) = Tu_x(0,t) - k_1u(0,t) - k_3u^3(0,t) - \alpha u_t(0,t) \tag{7}$$

will be considered, where k_1 and k_3 are constants. When the displacement $u(x,t)$ is small, or when k_3 is small, perturbation methods like the two time-scales perturbation method can be used to construct accurate approximations of the solution, which are valid on a long time scale.

Conclusions and future work

In this paper it has been shown that the formula of d'Alembert can be applied to a much larger class of problems than is generally assumed. This also implies that a larger class of weakly nonlinear problems for wave equations can be studied by means of characteristic coordinates (see also recent work in [5, 6]).

References

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