Analytical study of interfacial three dimensional gravity waves in presence of current

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Summary. Analytical study of short crested interfacial gravity waves propagating between two infinite fluids layers has been carried out. These waves are the simplest form of three dimensional waves and are obtained through a reflection of a 2D wave train into a vertical wall. The fluids have different densities and the upper is moving relative to the lower with horizontal and uniform velocity U. The fluids are taken to be incompressible and inviscid and the motion assumed to be irrotational. Using a perturbation method, the forth order solutions was obtained. Using this method attention was focused for the determination of the harmonic resonance of short-crested interfacial waves.

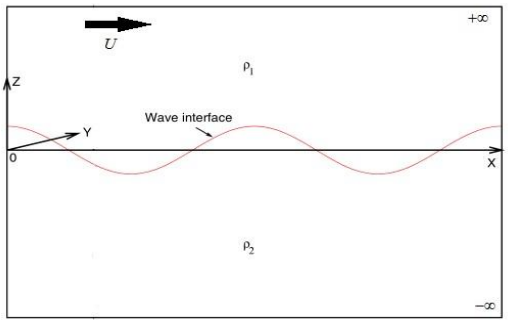
Introduction

For several years efforts have been devoted to the study of the two dimensional interfacial gravity waves in the presence of current. The most interesting approach to this issue has been proposed by Saffman [1], where he identified the critical current which limiting the existence of steady solutions. However, modeling some aspects of three dimensional interfacial gravity waves in the presence of current is essential for more realistic descriptions. Applying this modeling to appropriate cases of practical interest is the main goal of the present work. Most of the work published for three dimensional waves often deals with short crested waves which represent the simplest form, that is why, in our study we focused in this wave form. Using perturbation method an analytical approach has been presented up to the fourth order for the case of interfacial waves between two infinite fluid layers with different densities.

System of equations.

Considering Cartesian coordinate composed of the horizontal plane (x-y) of propagation's interfacial three dimensional wave between two infinite fluid layers (fig.1.a). Herein, short crested-wave field generated from the nonlinear interaction of two wave trains propagating to each other's having the same amplitude and wavelength. L is the wavelength of the incident wave, θ is the angle between the direction of incidence and the normal to the wall. U is the uniform current paralleling to the direction of propagation on the upper layer as shown in (fig.1.b).

(a)



(b)

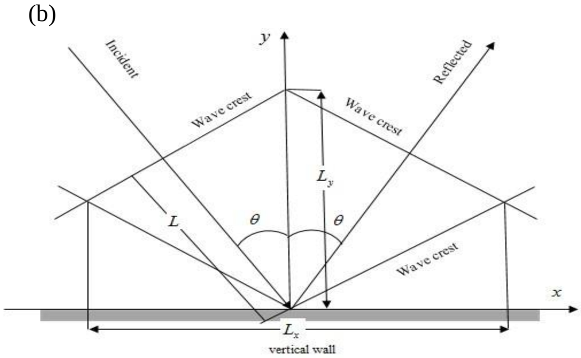


Figure.1: schematic presentation of a short crested wave in presence of current.

The two fluids with densities ρ_1 and ρ_2 are assumed to be inviscid, incompressible and the motion is irrotational. However, with this assumptions the fluid motion can be described by the velocity potential $\phi_1(X,Y,Z)$ and $\phi_2(X,Y,Z)$, which satisfy the dimensionless Laplace's equation in the two fluid layers domain.

$$\Delta\phi_1 = 0 \quad for \quad 0 < Z < +\infty, \tag{1}$$

$$\Delta\phi_2 = 0 \quad for \quad +\infty < Z < 0. \tag{2}$$

And due to the presence of a current on the upper layer, herein, the total velocity potential can be expressed by:

$$\phi_{1T}(X,Y,Z) = pUX + \phi_1(X,Y,Z). \tag{3}$$

Solutions of equations (1) and (2) are obtained using some boundary conditions for each layer of fluid, they are summarized in dimensionless form as below:

Kinematic boundary conditions at the interface: $(pU-\omega)\eta_X + p^2\eta_X\phi_{1X} + q^2\eta_Y\phi_{1Y} - \phi_{1Z} = 0,$ (4)

$$-\omega\eta_X + p^2\eta_X\phi_{2X} + q^2\eta_Y\phi_{2Y} - \phi_{2Z} = 0, \tag{5}$$

Dynamic boundary condition resulting from the Bernoulli's theorem where C is the Bernoulli's constant:

$$\mu\left\{(pU-\omega)\phi_{1X}+\eta+\frac{1}{2}(p^2\phi_{1X}^2+q^2\phi_{1Y}^2+\phi_{1Z}^2)\right\}+\omega\phi_{2X}-\eta-\frac{1}{2}(p^2\phi_{2X}^2+q^2\phi_{2Y}^2+\phi_{2Z}^2)+C=0, \tag{6}$$

Bottom boundary conditions: $\phi_{1Z}=0 \quad at \quad Z\to+\infty,$ (7)

$$\phi_{2Z}=0 \quad at \quad Z\to-\infty. \tag{8}$$

Perturbation method

The problem is solved by the perturbation method. Such an analysis assumes that the non linearities present a small corrections to linear wave theory. The parameters of the problem are developed in power series based on a small parameter h:

$$\phi=\sum_{r=1}^{\infty}h^r\phi_i^r(X,Y,Z),\ \ \eta=\sum_{r=1}^{\infty}h^r\eta_i^r(X,Y),\ \ \omega=\sum_{r=1}^{\infty}h^r\omega_r, \tag{9}$$

Where the wave steepness defined by $h=\frac{1}{2}\left(\eta\left(0,0\right)-\eta\left(0,\pi\right)\right)$.

In the first order, we retain only the linear terms. Thus, the linear solution is:

$$\begin{cases}\phi_1^{(1)}=(pU-\omega_0)\sin X\ \cos Y e^{-Z}, \\ \phi_2^{(1)}=\omega_0\sin X\ \cos Y e^{Z}, \\ \eta^{(1)}=\cos X\ \cos Y, \\ \omega_0=\mu pU+\sqrt{-\mu^2-\mu p^2U^2+1}\Big/\mu+1,\end{cases} \tag{10}$$

The dispersion relation for the linear interfacial wave can be expressed by: $\mu(pU-\omega_0)^2+\omega_0^2=1-\mu,$ (11)

From this relation the value of the critical current is given by: $U_{cl}=\sqrt{-\mu(\mu^2-1)}\Big/\mu p.$ (12)

As $U>U_{cl}$ the three-dimensional waves cannot be observed in regular form, in other words steady solutions no longer exist and the wave profile becomes unphysical.

In the higher order, the linearization of kinematic and dynamic equations is carried out by performing a Taylor series expansion of the potential velocities in the neighbourhood of $\eta=0$. After solving the system of equations, the following solutions are obtained in each order:

$$a_{mn}^{(r)}=-\frac{m(\mu pUf_{1,mn}^{(r)}-\mu\omega_0f_{1,mn}^{(r)}-\omega_0f_{2,mn}^{(r)})-\alpha_{mn}F_{mn}^{(r)}}{m^2(\mu p^2U^2-2\mu\omega_0pU+\mu\omega_0^2+\omega_0^2)+\alpha_{mn}(\mu-1)}, \tag{13}$$

$$b_{mn}^{(r)}=\frac{(pU-\omega_0)m^2\omega_0f_{2,mn}^{(r)}+(pU-\omega_0)mF_{mn}^{(r)}\alpha_{mn}+(m\omega_0+\mu\alpha_{mn}-\alpha_{mn})f_{1,mn}^{(r)}}{\alpha_{mn}\left[m^2(\mu p^2U^2-2\mu\omega_0pU+\mu\omega_0^2+\omega_0^2)+\alpha_{mn}(\mu-1)\right]}, \tag{14}$$

$$c_{mn}^{(r)}=-\frac{(\mu pU-\omega_0)m^2\omega_0f_{1,mn}^{(r)}-\omega_0mF_{mn}^{(r)}\alpha_{mn}+(p^2U^2+2pU\omega_{.0}+\omega_0^2)\mu m^2f_{2,mn}^{(r)}+(\mu-1)\alpha_{mn}f_{2,mn}^{(r)}}{\alpha_{mn}\left[m^2(\mu p^2U^2-2\mu\omega_0pU+\mu\omega_0^2+\omega_0^2)+\alpha_{mn}(\mu-1)\right]}, \tag{15}$$

A secular term appears when the denominator of the coefficients becomes zero. This condition is called harmonic resonance. The appearance of secular terms is characteristic of nonlinear short crested waves. Roberts [2] was the first to discover the phenomenon of harmonic resonance of short crested surface waves, and then extended by Allalou, Debiane and Kharif [3] for 3D interface waves in finite depth. When the resonance occurs, we have:

$$m^2(\mu p^2U^2-2\mu\omega_0pU+\mu\omega_0^2+\omega_0^2)+\alpha_{mn}(\mu-1)=0. \tag{16}$$

Relationship (16) is considered an extension of Allalou's, study in the presence of a uniform current.

Conclusions

In this work, we presented the internal three-dimensional gravity waves in the presence of a parallel current. The latter gave an original aspect to this study. Based on previous work mainly that carried out by N. Allalou & al. [3] it was found that our results were in good agreement. We presented all the physical aspects of the internal three-dimensional wave and we were able to extend our analytical results to the fourth order. We deduced the values of the critical velocity that did not exceeded throughout the study. Moreover, we were able to determine the neighbourhood of the harmonic resonance, and the observation of the wave profile was made by varying several parameters**.**

References

[1] SAFFMAN, P. G. & YUEN, H. C. 1982 Finite-amplitude interracial waves in the presence of a current.J. Fluid Mech. 123, 459-476.

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[3] N Allalou, M Debiane, and C Kharif. Three-dimensional periodic interfacial gravity waves: Analytical and numerical results, European Journal of Mechanics-B/Fluids, 371-386, 2011.