The Relevance of Spectral Submanifolds and Slow Manifolds for Randomly Excited Mechanical Systems

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Summary. Invariant manifolds, such as nonlinear normal modes, spectral submanifolds and normally hyperbolic invariant manifolds are the key to understand the dynamical behavior of nonlinear mechanical systems and serve as natural candidates for model-order reduction. While numerous related invariant manifold results are available for unforced and periodically forced nonlinear mechanical systems, the case of random external forcing has not been studied from this perspective. Here, we clarify the role of deterministic invariant manifolds in the case of small white noise excitation and demonstrate our results on explicit mechanical systems.

Introduction

To account for parameter uncertainty, unmodeled degrees of freedom or unknown disturbances in realistic engineering structures, the use of statistical methods is unavoidable [1]. Gaussian white noise is commonly considered to account for such random perturbations. Whether the deterministic invariant manifolds, such as nonlinear normal modes, spectral submanifolds or normally hyperbolic invariant manifolds, are of relevance under uncertainties or random external excitation has, however, remained unclear.

Various definitions of nonlinear normal modes have been proposed in the literature (cf. [2] for a review) and multiple computational algorithms for the unforced or periodically forced nonlinear mechanical systems have been developed [3]. Recently, Haller and Ponsioen [4] identified the smoothest nonlinear continuation of a spectral subspace of the linearization as spectral submanifold (SSM). While spectral submanifolds have proven valuable for model-order reduction, the related computational tools assume a completely deterministic nature of the system. This assumption, however, does not generally hold for structures which are subject to parameter uncertainties and external (unmodeled) disturbances.

Counterparts to deterministic dynamical features, such as normally hyperbolic invariant manifolds or stable/center manifolds of fixed points, for randomly perturbed mechanical systems are intricate to establish mathematically. Berglund and Gentz [5] assume an idealized slow-fast decomposition in the deterministic system, while the stable and center manifolds established in [6] generally depend on the realization of the random process. Therefore, each realization results in a different reduced-order model, which is computational costly and limits relevance of the reduced-order model significantly.

In contrast, the probability density function (PDF) is independent of the realization. In the case of Gaussian white noise excitation, the time evolution of the probability density is governed by the Fokker-Planck equation [7]. Explicit solutions of the Fokker-Planck equation, however, are only available in specific cases, and approximate numerical methods discretizing the Fokker-Planck equation or Monte Carlo simulations are computationally expensive [8].

Recently, Haller et al. [9] derived tools to identify material diffusion barriers, including material surfaces that extremize the diffusive transport of the PDF in the phase space. By the definition of these barriers, the transport of the PDF across them is purely driven the by small stochastic perturbations of the otherwise deterministic system. Specifically, *perfect barriers* block the transport at leading order completely and thereby demarcate regions of the phase space that trajectories generally not penetrate. Similarly, ridges of the diffusion barrier strength (*DBS*) highlight surfaces with strong diffusive transport, i.e., regions where trajectories accumulate. Both identifiers (perfect barriers and DBS) can be computed from purely deterministic quantities associated with the dynamical system and hence computationally expensive numerical methods such as Monte-Carlo approximations can be avoided.

Set-up

In this talk, we apply the methods from [9] for diffusive transport in fluids to mechanical systems excited by Gaussian white noise with small intensity. Specifically, we consider N-degree-of-freedom mechanical systems of the form

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} + \mathbf{S}(\mathbf{q}) = \sqrt{\nu}\mathbf{f}(\mathbf{q})d\mathbf{W}, \qquad \mathbf{q} \in \mathbb{R}^N, \tag{1}$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are the mass, damping and stiffness matrices. The geometric nonlinearity $\mathbf{S}(\mathbf{q})$ is a nonlinear function of the position, such that $\mathbf{S}(\mathbf{q}) = \mathcal{O}(|\mathbf{q}|^2)$. The vector $d\mathbf{W}$ collects M independent one-dimensional Brownian motions and the matrix $\mathbf{f}(\mathbf{q}) \in \mathbb{R}^{N\times M}$ prescribes their directions. Since $\mathbf{f}$ can depend on the coordinates $\mathbf{q}$ (cf. eq. (1)), our formulation also covers parametric random excitations.

After transforming system (1) into first order, we introduce the probability density function $\rho(\mathbf{q}, \dot{\mathbf{q}}, t)$ depending on the phase space location and time, both of which are suppressed in our notation for simplicity. The time evolution of ρ is then governed by the classic Fokker-Planck equation [7], which can be recast in the advection-diffusion form

$$\frac{D\rho}{Dt} = \frac{\nu}{2}\nabla \cdot \left(\nabla \cdot \mathbf{f}(\mathbf{q})\mathbf{f}^\top(\mathbf{q})\right) + k\rho. \tag{2}$$

The constant k in equation (2) depends on the damping $\mathbf{C}$ and $D\rho/Dt = \partial\rho/\partial t + \mathbf{v} \cdot \nabla\rho$ denotes the material derivative. The results of Haller et al. [9] on transport barriers can be applied to equations of the form (2). These results yield invaraint manifolds of an associated, deterministic barrier equation. The manifolds obtained in this fashion are barriers to the diffusive transport of the PDF of the original mechanical system (1).

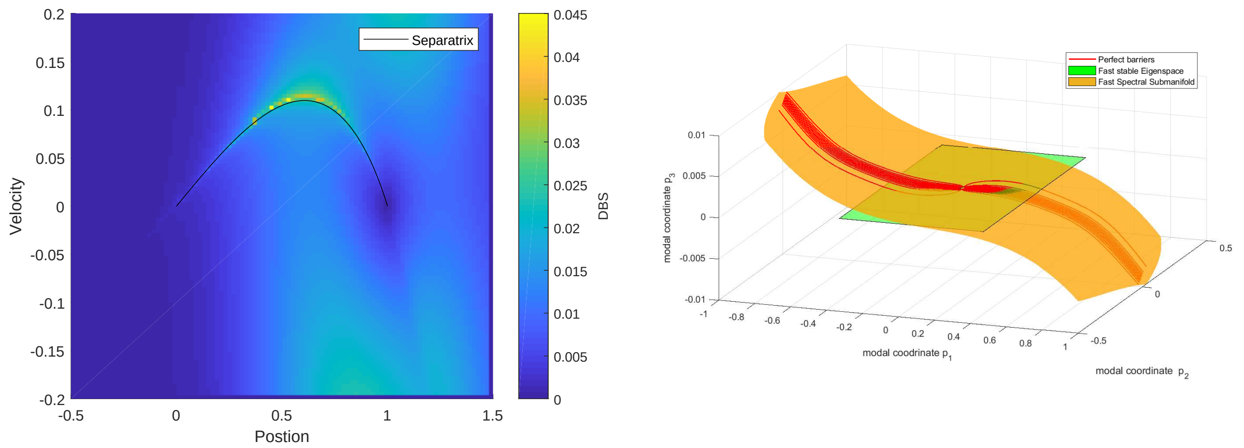
Results

We numerically investigate whether distinguished transport extremizers align with known deterministic slow manifolds or fast stable manifold of stable fixed points. More specifically, we investigate the modified Shaw-Pierre example [10]

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \ddot{\mathbf{q}} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_1 + c_2 \end{bmatrix} \dot{\mathbf{q}} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_1 + k_2 \end{bmatrix} \mathbf{q} + \begin{bmatrix} \kappa q_1^3 \\ 0 \end{bmatrix} = \sqrt{\nu} \begin{bmatrix} 0 \\ 1 \end{bmatrix} dw, \tag{3}$$

where the intensity ν is a small parameter ($\nu \ll 1$) and dw indicates Gaussian white noise. Further, we investigate the classical Duffing oscillator [11]

$$q + cq + kq + \kappa q^3 = \sqrt{\nu}\, dw. \tag{4}$$



(a) Duffing oscillator (4)

(b) Shaw-Pierre example (3).

Figure 1: Diagnostics from [9] applied to the Shaw-Pierre Example (3) and Duffing oscillator (4).

For systems (3) and (4), we plot the diagnostics for transport barriers from [9] in Fig. 1. In the case of the stochastically forced Duffing oscillator (4) the slow manifold connecting the saddle type fixed point at the origin with the stable node at $q = 1$ is successfully identified as a DBS ridge (cf. Fig. 1a). For the Shaw-Pierre example (3), the fast stable submanifold is successfully identified as a perfect barrier for the stochastically excited nonlinear system (cf. Fig. 1b).

Conclusion

We demonstrate that the methods developed in [9] link classic deterministic normally hyperbolic manifolds or stable submanifolds of nonlinear mechanical systems to the stochastic dynamics arising under small random perturbations to the deterministic system. Our analysis is based on purely deterministic quantities and thereby avoids computationally expensive methods, such as Monte-Carlo approximations.

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