Free propagation of nonlinear waves in 1D acoustic metamaterials with inertia amplification

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Summary. The free propagation of nonlinear harmonic waves in acoustic metamaterials with inertia amplification is investigated. A Lagrangian model is formulated to describe the nonlinear dynamics of a periodic chain of elastically coupled point masses (atoms), realizing a minimal 1D acoustic metamaterial with local inertia-amplifying oscillators. First, the nonlinear equations of motion governing the free undamped oscillations of the tetra-atomic periodic cell are formulated, and the linear dispersion properties governing the small-amplitude range of wave propagation are determined. Second, the harmonically-periodic solutions characterizing the high-amplitude range of wave oscillations are investigated, by employing the method of nonlinear maps. Some non-standard methodological tools are introduced to consistently apply the map approach to the implicit function characterizing the nonlinear difference equations.

Introduction

The band structure of microstructured periodic media has long been attracting the scientific interest of researchers in linear and nonlinear dynamics. In the last years, a renewed attention has been devoted to the parametric and computational design of phononic microstructured materials, targeted at fine-tuning the periodic microstructure to achieve unconventional, superior of functional dispersion properties [1]. In this respect, the pressing technological demand for light-weight materials serving as mechanical low-frequency filters or isolators has favoured the rapid diffusion and success of acoustic metamaterials [2]. Indeed, the free propagation of elastic waves in acoustic metamaterials can be inhibited – even in the absence of dissipation – by the linear mechanism of *local resonance*, which allows the opening, shifting and widening of spectral band gaps by properly tuning the natural frequency of auxiliary periodic oscillators (*resonators*), locally coupled to the cellular microstructure. From the physical viewpoint, low-frequency resonators tend to combine high flexibility with large inertial masses, conflicting with the requirement of material lightness. In order to circumvent this conundrum, proper solutions of inertia amplification can be adopted by introducing panthographic mechanisms, exploiting levered masses coupled in parallel with elastic stiffnesses [3]. In this framework of extreme mechanical solutions, the combination of high microstructural flexibility, pantographically-amplified oscillations and null or minimal dissipation can be the natural scenario for the development of important nonlinear dynamic phenomena.

Lagrangian model of the acoustic metamaterial

A Lagrangian model is formulated to describe the nonlinear dynamics of a periodic chain of undamped oscillators (Figure 1), in which only linear forces of attraction/repulsion are exchanged between any pair of adjacent point masses (*primary atoms*). The atomic chain represents a minimal physical realization of a 1D acoustic metamaterial with inertia-amplifying auxiliary oscillators (*secondary atoms*), rigidly connected to the primary atoms by a panthographic mechanism.

Equations of motion

Collecting the nondimensional displacement variables $u = U_1/L$, $w = (U_3 - U_1)/L$, $u_\ell = U_\ell/L$ and $u_r = U_r/L$ in the vector $\mathbf{u} = (u, w, u_\ell, u_r)$, the exact nonlinear equation governing the motion of the generic cell can be formulated. Expanding in Taylor series around the rest position $\mathbf{u} = \mathbf{0}$ and retaining terms up to the third order, the equation reads

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} + \mathbf{m}_2(\dot{\mathbf{u}}, \dot{\mathbf{u}}) + \mathbf{n}_2(\ddot{\mathbf{u}}, \mathbf{u}) + \mathbf{m}_3(\mathbf{u}, \dot{\mathbf{u}}, \dot{\mathbf{u}}) + \mathbf{n}_3(\ddot{\mathbf{u}}, \mathbf{u}, \mathbf{u}) = \mathbf{0} \quad (1)$$

The quasi-static equilibrium at the cell boundary nodes is instead governed by the linear equation $\mathbf{K}_p\mathbf{u} = \mathbf{f}_p$, where the external forces $\mathbf{f}_p = (f_\ell, f_r)$ can easily be related to the internal stresses $\boldsymbol{\sigma} = (\sigma_\ell, \sigma_r)$. Partitioning the displacement vector $\mathbf{u}$, the inner and outer displacement subvectors read $\mathbf{u}_a = (u, w)$ and $\mathbf{u}_p = (u_\ell, u_r)$, respectively. Collecting all the outer variables in the vector $\mathbf{v} = (u_\ell, u_r, \sigma_\ell, \sigma_r)$, the quasi-static equilibrium equations can be inverted to obtain the static condensation rule $\mathbf{u}_a = \mathbf{S}\mathbf{v}$. After condensation, the nonlinear equations of motion reads

$$\mathbf{M}\ddot{\mathbf{v}} + \mathbf{K}\mathbf{v} + \mathbf{p}_2(\dot{\mathbf{v}}, \dot{\mathbf{v}}) + \mathbf{q}_2(\ddot{\mathbf{v}}, \mathbf{v}) + \mathbf{p}_3(\mathbf{v}, \dot{\mathbf{v}}, \dot{\mathbf{v}}) + \mathbf{q}_3(\ddot{\mathbf{v}}, \mathbf{v}, \mathbf{v}) = \mathbf{0} \quad (2)$$

Focusing the analysis on the only periodic solutions in the nondimensional τ-time domain, the real-valued unknown $\mathbf{v}(\tau)$ can conveniently be expressed in Fourier series (truncated to account for the first harmonic terms)

$$\mathbf{v}(\tau) = \sum_{-\infty}^{\infty} \mathbf{a}_k \, \mathrm{e}^{\imath k\omega\tau} \simeq \mathbf{a}\, \mathrm{e}^{\imath\omega\tau} + \bar{\mathbf{a}}\, \mathrm{e}^{-\imath\omega\tau}, \qquad \text{with} \quad k \in \mathbb{Z} \quad (3)$$

where the Fourier coefficient $\mathbf{a} = (A_\ell, A_r, B_\ell, B_r)$ serves as (unknown) amplitude of the first harmonic component and bar indicates complex conjugate. The nondimensional parameter ω plays the role of circular frequency for the harmonic motion.

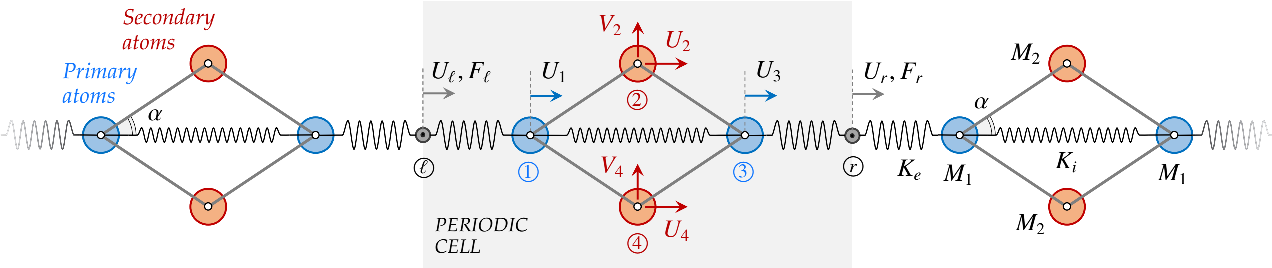

Figure 1: Acoustic metamaterial: (a) tetra-atomic crystal structure, (b) periodic cell of the lagrangian model, (c) mechanical properties.

Nonlinear map approach

The nonlinear equations of motion (2) can be linearized in the small-amplitude oscillation range. Therefore, the linear dispersion functions $\omega(\beta)$ relating the frequency ω to the nondimensional wavenumber β can be determined by applying either the Floquet-Bloch theory [4] or the map approach [5]. The latter technique employs the formal analogy between the wave periodicity (in the β-space) and the Lyapunov stability (in the τ-space) for discrete systems. The map approach can be applied to nonlinear systems governed by explicit equations $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y})$ to analyze the periodic solutions in the high-amplitude range of wave oscillations [6].

However, the nonlinear equations (2) can be manipulated to achieve only the implicit form $\mathbf{g}(\dot{\mathbf{y}}, \mathbf{y}) = \mathbf{0}$, which may require a different mathematical treatment [7]. Specifically, according to the most general definition of discrete implicit map, $\mathbf{y}_0$ is a p-periodic point of the implicit dynamic system $\boldsymbol{g}(\dot{\mathbf{y}}, \mathbf{y}, \boldsymbol{\alpha}) = \mathbf{0}$ if

$$\begin{cases} \boldsymbol{g}(\mathbf{y}_0, \mathbf{y}_1, \boldsymbol{\alpha}) = \mathbf{0} \\ \boldsymbol{g}(\mathbf{y}_1, \mathbf{y}_2, \boldsymbol{\alpha}) = \mathbf{0} \\ \dots \\ \boldsymbol{g}(\mathbf{y}_{p-2}, \mathbf{y}_{p-1}, \boldsymbol{\alpha}) = \mathbf{0} \\ \boldsymbol{g}(\mathbf{y}_{p-1}, \mathbf{y}_0, \boldsymbol{\alpha}) = \mathbf{0} \end{cases} \tag{4}$$

Therefore, the $\mathbf{y}_0$ stability can be analysed by introducing bifurcation conditions in order to assess the critical values of the parameter set $\boldsymbol{\alpha}$. The periodic points can be searched for the nonlinear system under investigation by setting $\dot{\mathbf{y}} = (A_r, B_r, \bar{A}_r, \bar{B}_r)$ and $\mathbf{y} = (A_\ell, B_\ell, \bar{A}_\ell, \bar{B}_\ell)$ and assuming ω as control parameter in the $\boldsymbol{\alpha}$-set.

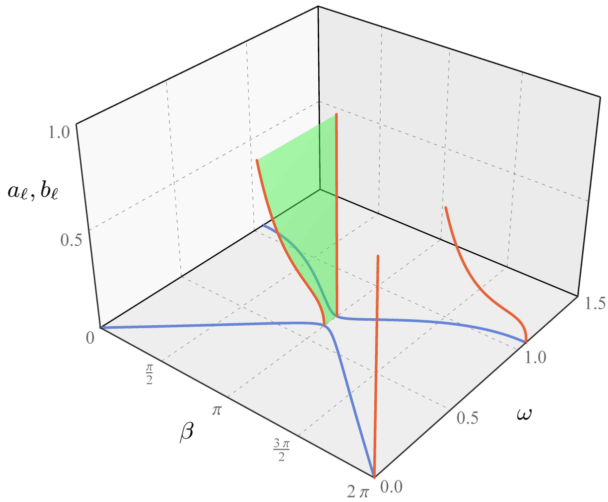

Figure 2: Linear spectrum (blue curves) and amplitude dependent frequency functions (red curves)

Equations (4) can be stated and solved for particular p-cases ($p = 1, 2$), giving solutions $\omega(\mathbf{y}_0, \mathbf{y}_1)$ corresponding to $\beta = 2\pi/p$. Using polar representations $A_\ell = a_\ell\, e^{\imath\phi}$ and $B_\ell = b_\ell\, e^{\imath\varphi}$, Figure 2 shows the amplitude-dependent frequency solutions $\omega(a_\ell)$ or $\omega(b_\ell)$, as obtained by setting $\mathbf{y}_1 = \mathbf{y}_0\, e^{\imath\, 2\pi/p}$. The amplitude-dependent frequency curves have dominant softening behaviour for small amplitudes, as expected for inertial nonlinearities [8], and originate from the linear spectrum for null amplitudes. The softening behaviour tends to enlarge the amplitude-dependent stop bandwidth (green region).

Conclusions

Nonlinear periodic solutions for the free wave propagation have been determined for a 1D acoustic metamaterial waveguide with inertia amplification. The nonlinear mapping approach has been employed, as it applies to discrete implicit maps. Amplitude-dependent frequency functions have been determined for particular oscillation periodicities.

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