Finding connecting orbits between saddle periodic orbits as organising centres of complicated dynamics

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Summary. We study heterodimensional cycles between two periodic orbits in a four-dimensional vector field. Such cycles are characterised by a connecting orbit that lies in the intersection of two two-dimensional manifolds; the returning connection is given by a family of connecting orbits in the generic two-dimensional intersection of two three-dimensional manifolds. Heterodimensional cycles are known to organise highly complicated dynamics, which persist under C^1-perturbations of the vector field. There are very few explicit examples known from applications; we study one such example, namely, a vector field model for calcium dynamics in a cell. We employ Lin's method to compute heterodimensional cycles and associated nearby global bifurcations. We present a cycle that is non-orientable and compute its locus in a two-parameter plane. In this way, we explore how it contributes to the organisation of the overall bifurcation structure, which, in turn, elucidates mechanisms behind the generation of C^1-robust chaotic dynamics.

A connecting cycle between two saddle periodic orbits is *heterodimensional* if the periodic orbits have unstable manifolds of different dimensions. Heterodimensional cycles can only exist in vector fields of dimension at least four, and are known to generate highly complicated dynamics [1, 2, 3, 5], including infinitely many periodic and/or homoclinic orbits. Furthermore, if a system has a codimension-one heterodimensional cycle, then every other system in a C^1-neighbourhood about the original system also has a heterodimensional cycle. We are interested in the mechanism behind such C^1-robustness, and study the existence and properties of heterodimensional cycles in an explicit four-dimensional vector field.

Heterodimensional cycles are primarily studied abstractly. In particular, there are very few known examples arising out of applications. We study a model for intracellular calcium oscillations that is known to feature a heterodimensional cycle [7]. The equations are given by

$$\left\{\begin{array}{rcl} \dot{c} &=& v, \\ D_c\,\dot{v} &=& s\,v - \left(\alpha + \dfrac{k_f\,c^2}{c^2+\varphi_1^2}\,n\right)\left(\dfrac{\gamma\,(c_t + D_c\,v - s\,c)}{s} - c\right) + k_s\,c - \delta\,(J - k_p\,c), \\ \dot{c_t} &=& \delta\,(J - k_p\,c), \\ s\,\dot{n} &=& \dfrac{1}{\tau}\left(\dfrac{\varphi_2}{\varphi_2 + c} - n\right). \end{array}\right. \tag{1}$$

Here, c represents the calcium concentration in the main part of the cell body (the cytosol) and c_t the total calcium concentration inside the cell (including that in an internal calcium store, known as the endoplasmic reticulum or ER); v is the membrane potential; and n is a gating variable that represents the fraction of open channels through which calcium enters the cytosol from the ER. System (1) is written in a moving-frame coordinate system and the differentiation is with respect to the travelling-wave coordinate. We choose the same parameters as in [7] with $s = 9.0$ fixed, and we vary the flux J of calcium entering from outside the cell as our bifurcation parameter; see Table 1.

α	k_s	k_f	k_p	φ_1	φ_2	τ	γ	D_c	δ	s
0.05	20.0	20.0	20.0	2.0	1.0	2.0	5.0	25.0	0.2	9.0

Table 1: Parameter values for the intracellular calcium model (1).

With this explicit model, we can leverage advanced numerical methods to study heterodimensional cycles in a concrete setting. To this end, we set up a two-point boundary value problem (2PBVP) based on Lin's method that represents the (non-robust) connecting orbit. More precisely, we define an orbit segment that starts near a saddle periodic orbit and ends in a three-dimensional cross-section Σ, which we choose to be locally transverse to the flow [6]. We also define a second orbit segment that starts in Σ and ends near another saddle periodic orbit. Here, we fix parameters as in Table 1 and start with $J = 2.957$, which we estimate to be close to the actual bifurcation value. The two orbit segments are restricted such that the difference between their end points in Σ lies in a prescribed *Lin direction*. This 2PBVP can be solved by pseudo-arclength continuation techniques in AUTO [4]. As we vary J, we detect the connecting orbit as a zero of the distance between the end points in Σ. Once the connecting orbit has been found, its locus can be computed by varying two system parameters while keeping the distance at zero.

We find a heterodimensional cycle in system (1) for $J \approx 2.95748$; this cycle is shown in Figure 1 in projection onto (c, v, c_t)-space. Specifically, we show the single connecting orbit Ω_1^{PD} (blue) from the saddle periodic orbit Γ_1 (green) to the saddle periodic orbit Γ_{PD} (green), and the family S of orbits from Γ_{PD} to Γ_1 that forms a two-dimensional surface (red).

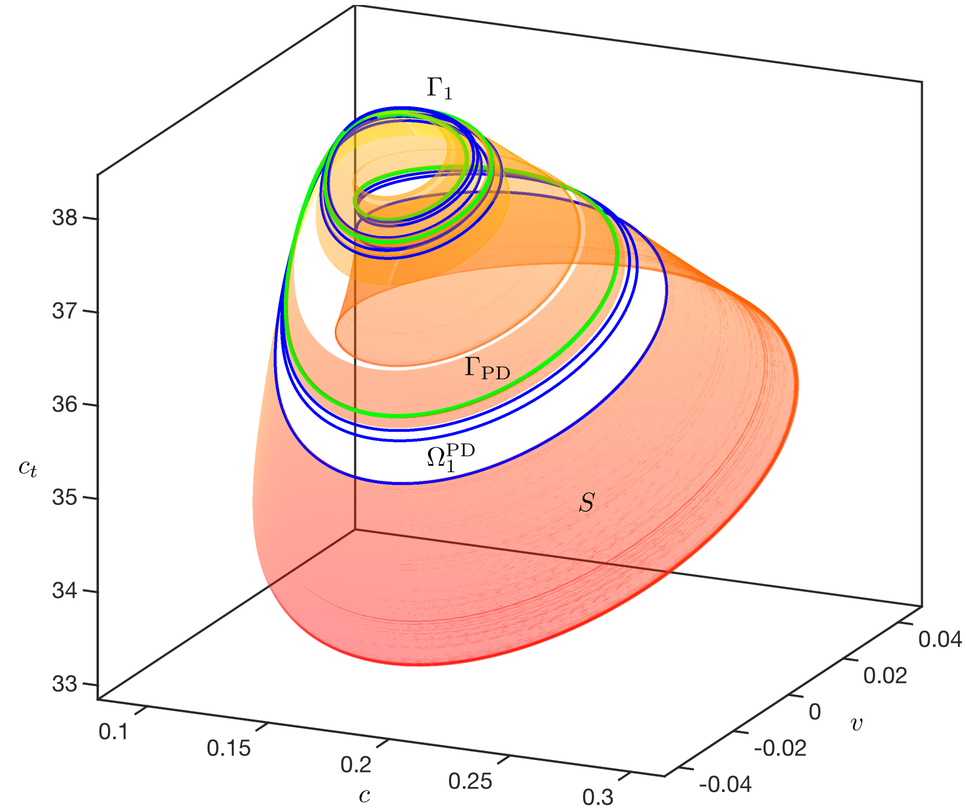

Figure 1: Three-dimensional projection onto (c, v, c_t)-space of the non-orientable heterodimensional cycle in system (1) with $J \approx 2.95748$. Shown are saddle periodic orbits (green) Γ_1 and $\Gamma_{\rm PD}$, the codimension-one connecting orbit $\Omega_1^{\rm PD}$ (blue) from Γ_1 to $\Gamma_{\rm PD}$, and the returning surface S (red) of robust connections.

Importantly, the heterodimensional cycle in Figure 1 is different from the one found in [7]. Its distinguishing feature is that Γ_1 is non-orientable. As we decrease J from near the Hopf bifurcation that creates Γ_1, this periodic orbit undergoes a period-doubling bifurcation, so that it has one negative unstable, one negative stable, and one positive stable Floquet multiplier, with the positive one being associated with the strongest contracting direction. This period-doubling bifurcation is subcritical, and gives rise to a period-doubled orbit that undergoes its own period-doubling bifurcation before merging with $\Gamma_{\rm PD}$ at a fold bifurcation. The periodic orbit $\Gamma_{\rm PD}$ in Figure 1 has one negative stable, one negative unstable, and one positive unstable Floquet multiplier. The closure of S is non-orientable since it is tangent to the weakly stable linear bundle of Γ_1, which is associated with the negative stable Floquet multiplier. The non-orientability complicates the geometry of S: S accumulates onto Γ_1 in a twisting fashion, since the weakly stable linear bundle of Γ_1 locally forms a Möbius band.

Starting from the heterodimensional cycle for $J \approx 2.95748$, we can compute the one-parameter family of this cycle in a two-parameter plane. By studying the bifurcation structure of system (1) locally about this locus, we examine how the parameter plane is organised to give rise to open regions where heterodimensional cycles are found robustly.

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