An extreme time-periodic oscillator.

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Summary. Parametric instabilities are dynamical instabilities possibly arising when the mechanical state of a structure is periodically modulated in time. It is sometimes seen as a phenomenon to avoid for example with sailing ships (parametric rolling) but it can also be exploited to study vibrating fluids (Faraday waves [1]) or Nano-Electro- Mechanical Systems [2]. One well-known limitation in fully exploiting classic parametric instabilities based on small periodic modulation of a mechanical state is that inherent friction forces rapidly cancel sub harmonic parametric resonances. To overcome this drawback, we propose to "extremely" modulate the mechanical state of the system in order to enhance parametric instabilities and therefore allow for new promising dynamic functionalities. This original way of enhancing and controlling parametric instabilities is illustrated here through the numerical and experimental implementation of an electromagnetic pendulum. We find that it is possible to greatly enhance the number of sub harmonic instability regions and also that the width of these regions can be controlled.
Keywords : Parametric instabilities, structural vibrations, experimental vibrations, Meissner Equation, Floquet theory.

Experimental system under study

Our goal is to periodically vary a mechanical system between two very different states in order to enhance parametric instabilities, even in the presence of classic internal friction forces. For illustrative purposes, we set up in the lab the proof of concept shown in Fig.1a). The experiment consists of a magnetic pendulum that is symmetrically placed between two attracting electromagnets. When the electromagnets are off, the system is a simple pendulum characterized by a natural frequency $\omega_0 \approx 9$ rad/s as illustrated in the experimental plot of Fig.1c). When turning the electromagnets on through an electrical current I, the state of the pendulum can be drastically modified. In our example of Fig.1, when the control parameter I is slightly below I_{max}, our system is naturally oscillating with a slower natural frequency. Above $I = I_{max}$, our system is no more oscillating but diverging: attracted to the right or left electromagnet depending on the imperfections in our experiment. This mechanical system is therefore a simple first realization of what we coined an *extreme parametric oscillator*: with a single parameter, in this case I, we are drastically and easily changing the state of our system. In classical parametric oscillators, this extreme modulation is hardly reachable because the geometrical mechanical modulation parameters that come into play (the length of the pendulum or the effective gravity for example) are not easily varied on such scales [1][2].

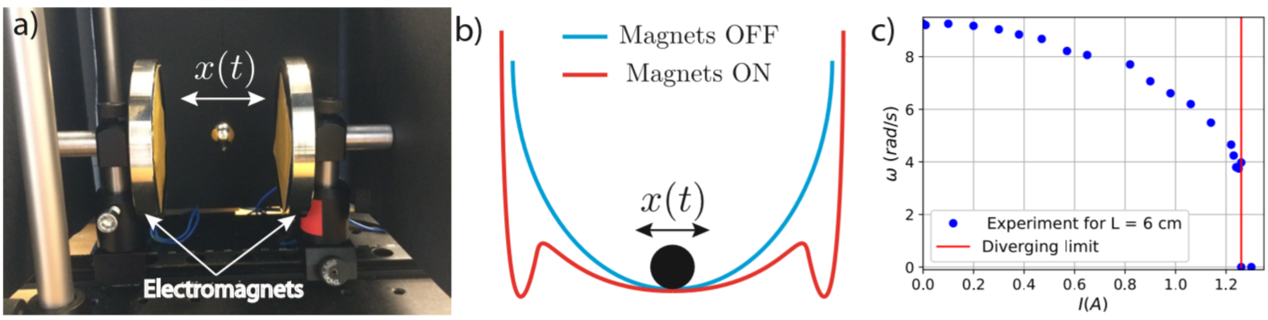

Figure 1: The electromagnetic parametric oscillator under study. a) A pendulum whose mass is made of steel, is symmetrically placed between two identical attracting electromagnets that are periodically turned on (red energy states in b)) and off (blue energy states in b)). b) Simplified "Particle in a time-varying potential well" model. c) Evolution of the natural frequency of the pendulum for various value of the electrical current I in the two electromagnets when the laters are separated by $L = 6$ cm. Below $I < I_{max}$, the pendulum is naturally oscillating if perturbed. Eventually, for I close to the diverging limit I_{max} , the natural frequency goes down. Above I_{max} the mass is no more oscillating but diverging.

To investigate the dynamic behavior of our extreme parametric oscillator, we periodically turn the electromagnets on or off with a period T, in a square wave fashion making the system modulate bewteen two states (see Fig.2a)). Fig.2b) represents the experimental stability diagram of the pendulum in the modulation parameter space (T, I). For some (T, I), the system is dynamically stable (blue triangles), i.e. the pendulum is slightly vibrating (because of small imperfections) but stays close to the trivial vertical state. The crosses indicate the modulation parameters for which the mass was parametrically unstable, i.e. dynamically impacting the electromagnets. The color legend indicates the number of cycles the pendulum is doing in the emerging nonlinear vibrational regime. Modes with an integer number of M represent T-periodic unstable regions when the other M numbers represent $2T$ unstable regions. At relative low modulation amplitude, $I < 1$ A, the pendulum is often stable, except eventually for the first or second parametric instability regions. For "extreme" modulation amplitude such as $I \approx I_{max}$, it is possible to trigger highly sub-harmonic instability regions, here up to the 58^{th} instability region ($M = 29$) when the current record demonstrated in a micro electromechanical device is found to be the 28^{th} instability region [2].

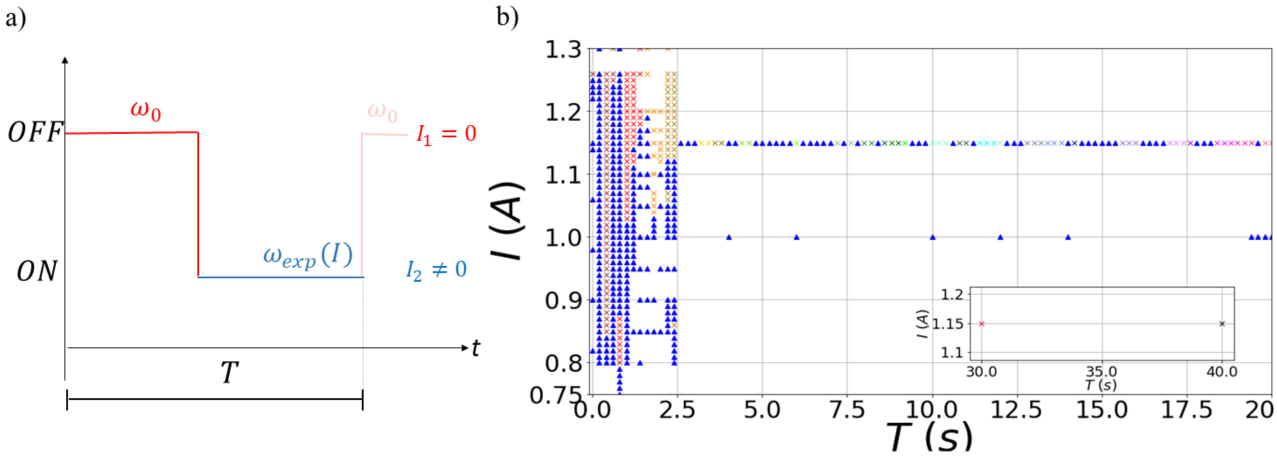

Figure 2: a) The electromagnets are turned on and off in a square wave fashion with period T. The amplitude of modulation is the electrical current I when the electromagnets are on. b) Experimental stability chart of the pendulum in the modulation parameter space (T, I). Blue triangles represent stable states where the mass stays close to the middle of the electromagnets. Crosses represent unstable states where the pendulum dynamically diverges and eventually impact the electromagnets.

Model of an extreme time-periodic oscillator

The aforementioned experimental mechanical system can be seen as a non-damped pendulum that oscillates periodically between a natural frequency ω_0 and ω_{exp} following a square wave function. Based on the theoretical study of the Meissner equation[3] a linear theoretical model for the stability of the vertical pendum can be obtained:

$$\ddot{\theta}(\tau) + \left(\alpha^2 + \beta^2 sgn(cos(\tau))\right)\theta(\tau) = 0 \tag{1}$$

with α^2 and β^2 two dimensionless parameters. Fig.3 represents the stability of Fig.2b) represented in the analytical space (α^2, β^2). With this experimental setup we are able to observe extreme periodic instability for the first time (large values for α^2 and β^2). For small values of α^2 and β^2 the model represents correctly the evolution of the experimental system. The alternation between stable and unstable analytical regions shows a good agreement with those found experimentally. In conclusion it is possible to trigger extreme parametric oscillations experimentally and to develop a corresponding linear theoretical model. This new approach could be promising for very large-band energy harvesting devices.

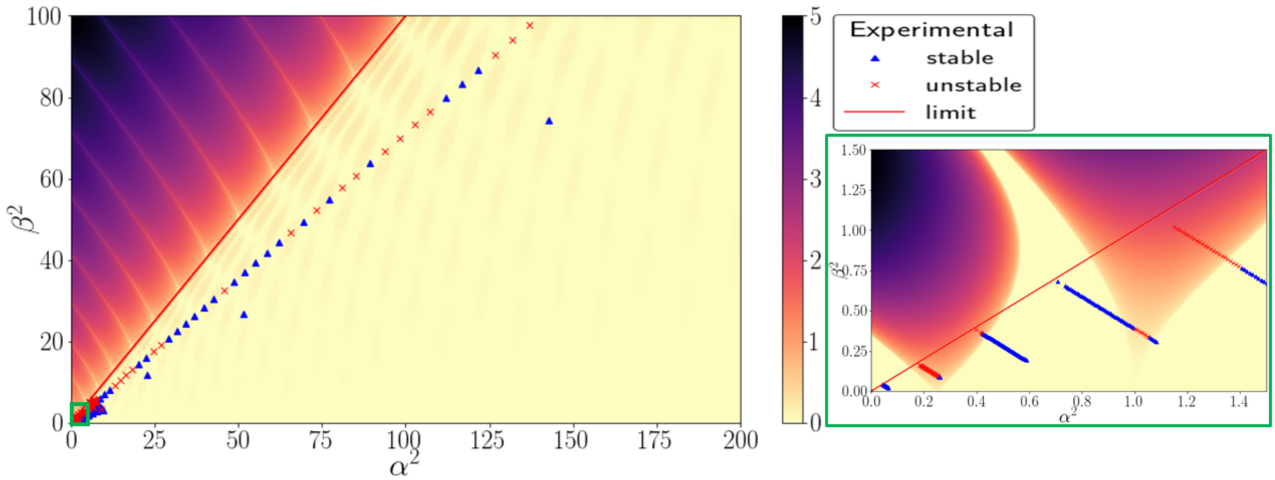

Figure 3: Analytical stability chart with experimental results presenting several instability regions. The numerical results represent the evolution of the real part of the Floquet exponent of equation (1) [4]. If it is equal to zero than the movement of the system is stable. Values larger than zero mean the solution of the system increases exponentially so the system is unstable. The blue triangles correspond to the experimental stable states and the red crosses correspond to the unstable ones. The red curve ($\alpha^2 = \beta^2$) represents the limit between systems that are naturally stable and systems naturally unstable (over the limit).

References

[1] S. Douady (1990) Experimental study of the Faraday instability *J. Fluid Mech* ,Vol 221, pp383-409.

[2] Y. Jia, S. Du and A.A. Seshia (2016) Twenty-Eight Orders of Parametric Resonance in a Microelectromechanical Device for Multi-band Vibration Energy Harvesting. *Scientific Reports* Vol 6, 30167.

[3] Chikara Sato (2015) Correction of Stability Curves in Hill-Meissner's Equation. *International Centre for Mechanical Sciences* Vol 562.

[4] Sébastien Neukirch, Arnaud Lazarus and Corrado Maurini (2015) Stablility of discretized nonlinear elastic systems. *International Centre for Mechanical Sciences* Vol 562.