Playing dominoes with lasers: Excitability and pulsed solutions of the Yamada model for a semiconductor laser with saturable absorber and delayed optical feedback

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Summary. We are concerned with the generation of periodic light pulses in semiconductor lasers with saturable absorber through delayed re-injection of the pulse via a reflecting mirror. More specifically, we consider the Yamada model with delayed feedback and study the generation of periodic pulse trains via repeated self-excitation after passage through the delayed feedback loop. We employ numerical continuation and analytical methods for delay differential equations to characterize these periodic solution and their bifurcations.

Surely everyone is familiar with chain reactions of falling domino tiles. After careful preparation of upright rectangular tiles along some path and in close proximity to one another, toppling the first tile leads to a chain reaction that hopefully causes all dominoes to fall. Such setups shall serve as an analogy for directed networks of delay-coupled excitable systems. Here, a sufficiently large perturbation (toppling the first tile) is capable of producing an excitation, in the form of a pulse, a spike or some kind of localized change in a physical quantity (momentum in the case of dominoes), which propagates to a subsequent node in the network and itself serves as a perturbation. If this perturbation is again large enough, a secondary excitation is created and the process continues. The speed of this propagation is finite and can be modeled via time delays between the nodes. Specific examples of this kind of relay are networks of semiconductor lasers with saturable absorber; another is the human brain consisting of millions of connected neurons, which are hard to study both experimentally and analytically.
Motivated by experiments with a micropillar semiconductor laser with saturable absorber and a reflecting mirror [1], we are concerned with the simplest network motif allowing for sustained re-excitation: a single node with delayed self-coupling. We modify the Yamada model [2] for an excitable or self-pulsating laser to incorporate delayed optical self-feedback [3]. Although seemingly simple, this setup allows for the generation of periodic light pulses that constitute the basis for modern telecommunication, material processing and high energy physics.
Here, we present an analytical tool that characterizes the onset and termination of such light pulses as a bifurcation in this system of delay differential equations (DDE). In dimensionless form, the model reads

$$G'(t) = \gamma\,(A - G(t)[1 + I(t)])\,, \tag{1}$$
$$Q'(t) = \gamma\,(B - Q(t)[1 + aI(t)])\,, \tag{2}$$
$$I'(t) = [G(t) - Q(t) - 1]\,I(t) + \kappa I(t-\tau). \tag{3}$$

Equations (1)–(3) describe the time evolution of the laser output intensity $I(t)$, as well as the gain $G(t)$ and absorption $Q(t)$ of photons in the laser cavity. The parameter A is the pump strength (the amount of energy provided to the laser), $B, a, \gamma > 0$ describe physical properties of the laser; and they differ from device to device. The time delay τ reflects the distance from the mirror to the laser cavity and the feedback strength κ accounts for the losses along that feedback loop. We fix parameters $A = 6.5, B = 5.8, a = 1.8$, and $\gamma = 0.04$ as in [4] and refer the interested reader to [1, 2] for the physical details.
The excitability of Eqs. (1)–(3) is governed by a homoclinic bifurcation [4]. For the considered parameter values, the stable manifold of a steady state locally separates the state space of the system [5]. The laser is essentially off, yet a sufficiently large perturbation that brings the system above this manifold, causes a fast increase in I that is followed by fast relaxation. Such an orbit is referred to as a pulse in the context of lasers; see Fig. 1(a, black) for an example. At the homoclinic bifurcation, the stable and unstable manifold of the equilibrium intersect transversally and give rise to a periodic orbit [4].
In the following, we focus on the full system (1)–(3) and study the effects of delayed feedback to the system in the excitable configuration. A detailed bifurcation analysis of Eqs. (1)–(3) has been carried out earlier by some of the authors [6], and we present here a refined analysis with regard to periodic orbits that are generated by sustained self-excitation after one delayed feedback loop. To illustrate our results, we fix $\tau = 5000$ and $\kappa = 0.52$ and prepare the pulse in Fig. 1(a) as the initial condition for system (1)–(3). We observe sustained pulsation: each pulse travels along the delayed feedback loop and triggers a subsequent pulse after approximately one delay interval. Remarkably, the pulsed solution reached from this initial condition has a higher peak intensity than the original pulse of the solitary system, and we observe a different profile; see Fig. 1(b, blue). Our formal analysis reveals that these solutions correspond to different periodic orbits, which coexist for the chosen parameter values. In fact, there is a third type of small pulsed solution present for the chosen parameter values; see Fig. 1(c, red).
We employ numerical continuation to reveal that these solutions correspond to different branches (black, blue, red) of periodic orbits when the time delay is varied as a parameter; see Fig. 1(d). The branches of periodic orbits share a

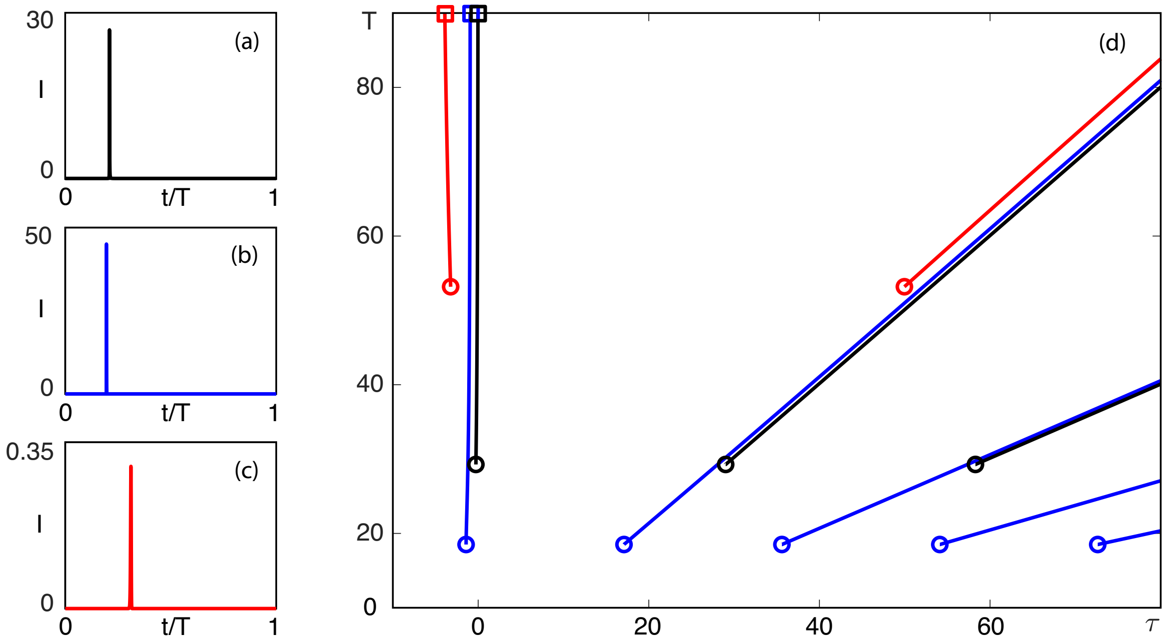

Figure 1: Panels (a)-(c) show the I-component of coexisting pulsed solutions with large period for large delay $\tau = 5000$. Panel (d) is the corresponding bifurcation diagram for small values of the delay; shown are the period of the periodic orbits (solid curves), points of Hopf bifurcations (circles), and indicated loci of Homoclinic bifurcations (squares). Other parameters are $A = 6.5, B = 5.8, a = 1.8, \gamma = 0.04$, and $\kappa = 0.52$.

remarkable property: their period scales effectively linearly with the delay. More precisely, the period T along a given family satisfies $T = \tau + \delta$, where $\delta \ll \tau$ and $\delta/\tau = T/\tau - 1 \to 0$ as $\tau \to \infty$. The intuition is clear: the I-component of the pulsed solutions is close to zero throughout most of their period. Therefore, a pulse that is emitted at time t, travels the length of the delay loop and only affects the system at time $t + \tau$. Additionally, there is a certain response time δ of the system to build up in intensity and to produce a subsequent pulse. Such periodic solutions are also referred to as Temporal Dissipative Solitons (TDS) in DDEs [7]. Each branch of pulsed periodic orbits in system (1)–(3) corresponds to a family of periodic solutions of the so-called profile equation, given by Eqs. (1)–(2) and

$$I'(t) = [G(t) - Q(t) - 1]\, I(t) + \kappa I(t + \delta), \tag{4}$$

which undergoes a homoclinic bifurcation. Figure 1(d) shows that for each branch of periodic orbits (black, blue, black) with $T > \tau$, for $\tau > 0$, there is a branch (red, blue, black) of periodic orbits with $\tau < 0$ exhibiting a homoclinic bifurcation. This correspondence can be seen by using the concept of reappearance of periodic orbits for DDEs [8].

Hence, we study the pulsed solutions of Eqs. (1)–(3) indirectly via bifurcations and codimension-two points of homoclinic orbits in Eqs. (1),(2) and (4). We show that the onset and termination of such pulse trains correspond to a bifurcation of countably many saddle-node periodic orbits with infinite period; moreover, we show these bifurcations coincide with codimension-two points along the family of homoclinic orbits as κ is varied as a parameter. This approach allows us to compute the corresponding critical coupling strengths that give rise to pulsed solutions in Eqs. (1)-(3). Our methodology is relevant for the rigorous analysis of delay-coupled excitable systems in general. Future applications will include recurrent networks of lasers and neurons, which are in a similar excitable configuration when at rest.

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