# Bifurcations in Piecewise Smooth, Delay Differential Systems

Amit Shukla*, N.W.A. Strijbosch† and Henk Nijmeijer †
**Department of Mechanical and Manufacturing Engineering, Miami University, Oxford, Ohio, USA*
*†Department of Mechanical Engineering, Eindhoven University of Technology, The Netherlands*

*Summary*. This paper presents bifurcation study of piecewise smooth differential delay equations (PWS-DDEs) where the state variable determines which differential delay equation is active. This paper aims to add to the existing research performed on bifurcation studies of PWS-DDEs by developing an algorithm to perform bifurcation studies of a general form of PWS-DDEs, such that it can be used for any bifurcation study of a PWS-DDEs system. Bifurcation studies are performed using a simplified model of a controlled inverted pendulum to study the effect of parameters. During the bifurcation studies, new type of bifurcations that arise for PWS-DDEs are defined.

## Introduction

The ability to predict the stability of periodic behavior can be of importance in the research into for instance Parkinson [2] or other balance disorders. Methods to determine periodic solutions for differential delay equations (DDEs) or piecewise smooth (PWS) systems are well described. To find periodic solutions of DDEs the collocation method is often used [3, 4, 5, 6]. In [1] the collocation method as used for DDEs is extended such that it is able to find periodic solutions of PWS-DDEs.This paper presents bifurcation studies of a general form of PWS-DDEs. We study the influence of smooth switching manifold on the parametric bifurcations for a controlled inverted pendulum. Bifurcation studies are presented for systems with smooth switching manifolds using the algorithm developed.

## Model

The state space model of the inverted pendulum is given by

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}x_2(t)\\ \sin x_1(t) - x_2(t) - u(x(t-\tau))\end{bmatrix} \tag{1}$$

where $x_1 := \theta$ and $x_2 := \dot{\theta}$, where $\theta$ is the angle and $u$ is a torque determined by a controller. Delays in for instance the sensors or computational delays are modeled by using the delayed values of the angle and angular velocity to determine the torque $u$.The parameters (mass, length, and damping coefficient) in this model are scaled to unity.Some studies suggest that central nervous system uses control switching strategies that deactivate the controller in certain (stable) subspaces of the state space, to minimize control effort [7]. The switching strategies are a function of these delayed values, hence one way to describe the control input on the system can be:

$$u(x(t-\tau)) = \begin{cases}0 & \text{when } x(t-\tau)\in\chi_{unctrl}\\ K_p x_1(t-\tau) + K_d x_2(t-\tau) & \text{when } x(t-\tau)\in\chi_{ctrl}\end{cases} \tag{2}$$

with $\chi_{unctrl}\bigcup\chi_{ctrl} = \mathbb{R}^2$, with $K_p$ and $K_d$ the proportional and derivative gain, respectively. The switching strategy studied here is to only control the system if magnitude of the angle is large. The subspaces $\chi_{unctrl} = \{x\in\mathbb{R}^2|\,|x_1|\le e\}$ and $\chi_{ctrl} = \{x\in\mathbb{R}^2|\,|x_1| > e\}$ define this switching strategy where $e$ is the value of $|\theta(t-\tau)|$ which defines the switching plane.

## Bifurcation studies

A bifurcation study along parameter $K_p$ is performed to study the effect of switching manifold. The bifurcation behavior is characterized by appearance of topologically nonequivalent phase portraits under variation of the parameter $\mu$. The bifurcation studies are performed using the switching plane with $e = 0.1$. The parameters of the model are $K_p = 2$, $K_d = 1$ and $\tau = 0.5$, if not mentioned differently. The bifurcation diagram along the bifurcation parameter $K_p$ is depicted in Figure 1. The whole branch consist of stable periodic solutions and the unstable equilibrium point at the origin. The evolution of a branch at $K_p = 1$ is given in Figure 2. Due to the sudden periodic solutions that originate similar to a Pitchfork bifurcation this phenomena will be referred to as the *Pseudo-pitchfork bifurcation*. There are two differences between pseudo-pitchfork bifurcation and the pitchfork bifurcation:the periodic solutions do not originate from the main branch and stability along the main branch does not change at the bifurcation point. For phase portrait $C$ in Figure 1 one side has a stable periodic solution, and the other side there are two stable periodic solutions and on each side of the bifurcation point the solutions are stable. To study the behavior that causes the bifurcation, a section of the whole branch around the bifurcation point is depicted in Figure 3. In this case, for $\mu < \mu_0$, with $\mu_0$ the bifurcation point, there is one unstable equilibrium one periodic solution, and for $\mu \ge \mu_0$ there is still one unstable equilibrium and two stable periodic solutions. This is called an *X-bifurcation*. The most challenging part of these studies is an inherent inability to capture all bifurcations at any critical point due to an inherent numerical approach and this remains part of future work for authors.

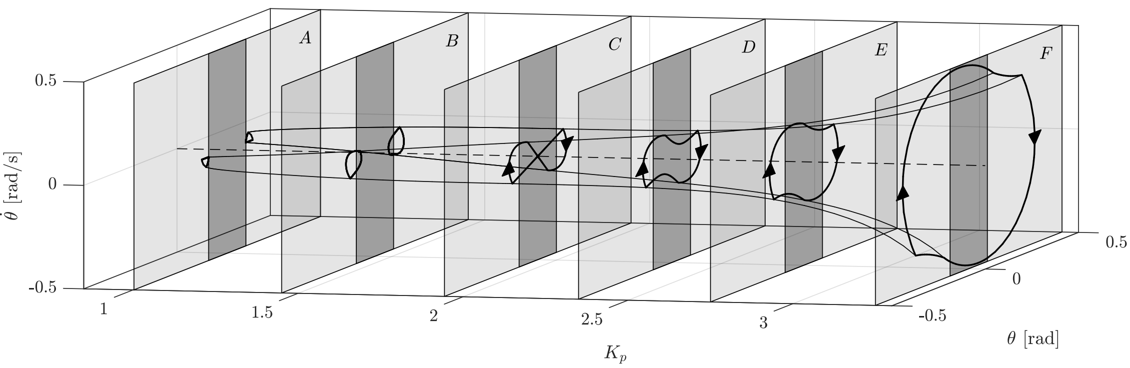


Figure 1: Bifurcation diagram along parameter $K_p$ with the magnitude switching manifold

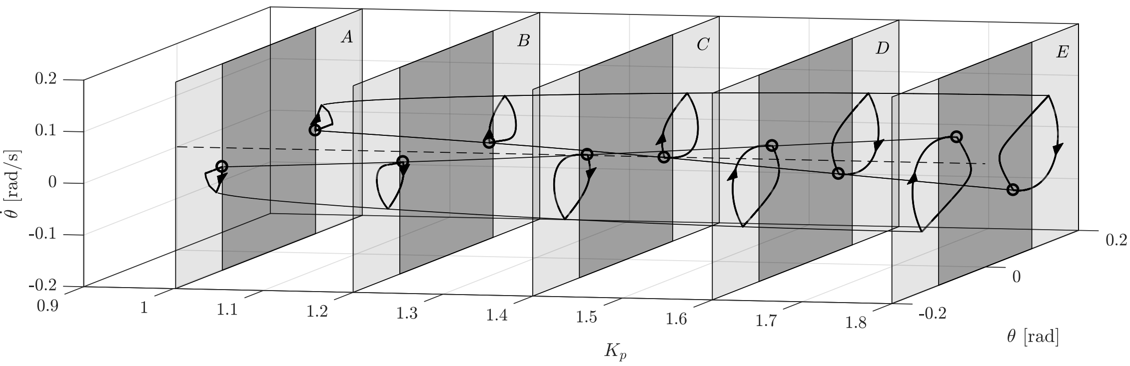


Figure 2: Evolution of periodic solutions close to Pseudo-pitchfork bifurcation

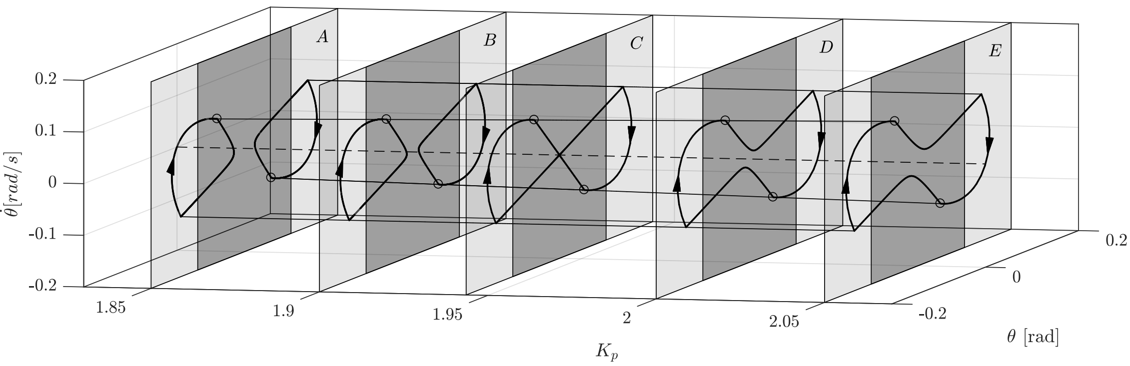


Figure 3: Evolution of periodic solutions close to X-bifurcation

## Conclusions

Two new bifurcations are observed for the system using a smooth switching control strategy for a PWS-DDE system.

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