Feedback Control for a Body Carrying a Chain of Oscillators

Igor Ananievski and Alexander Ovseevich
Ishlinsky Institute for Problems in Mechanics RAS
Prospekt Vernadskogo, 101-1, Moscow 119526 Russia

Summary. We study steering of a linear chain of point masses connected via springs to the equilibrium, by means of a bounded force applied to the first mass in the chain. In other words, we design an explicit feedback control that bring the system to a given terminal rest state in a finite time. Thus, we prove the complete controllability of the system, and describe explicitly the feedback control proposed. Then, we show its robustness with respect to unknown disturbances.

The chain of n oscillators: Problem statement

We consider a control problem for a system representing a solid body carrying a chain of n linear oscillators modelled by point masses connected via springs (Fig. 1). The whole system moves along a horizontal line under the action of a control force and an external disturbance applied to the carrying body.

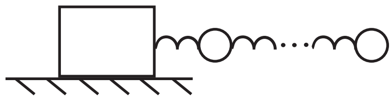

Figure 1: A body carrying the chain of oscillators

Equations of motion have the form

$$\begin{aligned} &m_0\ddot{x}_0 = -k_1x_0 + k_1x_1 + u + v \\ &m_i\ddot{x}_i = k_ix_{i-1} - (k_i + k_{i+1})x_i + k_{i+1}x_{i+1}, \quad i = 1, \dots, n-1 \\ &m_n\ddot{x}_n = -k_nx_{n-1} + k_nx_n \end{aligned} \tag{1}$$

Here, we have the controllable mass m_0, which is subject to control u, and disturbances v. Each mass $m_i, i = 0, \dots, n$ has coordinate x_i, and the Hooke law $F_i = k_i(x_i - x_{i-1})$ gives the force from mass m_{i-1} to m_i.
Assuming $x = (x_0, x_1, \dots, x_n)$, we can write (1) in the Cauchy normal form

$$\dot{x} = A_1x + \frac{1}{m_0}b(u + v)$$

where A and b are the block matrices

$$A_1 = \begin{pmatrix} 0 & I \\ A_0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ e \end{pmatrix}$$

$$A_0 = \begin{pmatrix} -\dfrac{k_1}{m_0} & \dfrac{k_1}{m_0} & 0 & \dots & 0 \\ \dfrac{k_1}{m_1} & -\dfrac{k_1+k_2}{m_1} & \dfrac{k_2}{m_1} & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \dfrac{k_{n-1}}{m_{n-1}} & -\dfrac{k_{n-1}+k_n}{m_{n-1}} & \dfrac{k_n}{m_{n-1}} \\ 0 & \dots & 0 & \dfrac{k_n}{m_n} & -\dfrac{k_n}{m_n} \end{pmatrix}, \ e = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

and I is the unit matrix.
We assume that we know the current values $x_0, \dot{x}_0, x_1$, while the other coordinates are not directly observable. Therefore, a measured output is

$$y = Dx, \quad D = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots 0 \\ 0 & 1 & 0 & 0 & \dots 0 \\ 0 & 0 & 1 & 0 & \dots 0 \end{pmatrix}$$

Theorem 1 *Pair (A_1, b) is controllable and the pair (A_1, D) is observable.*

Thus, we can in principle bring the system under consideration to any given state, at least if the disturbances are absent.

Control of canonical systems

The control problem stated can be reduced [1, 2] to the control problem for canonical system

$$\dot{z} = Az + B(u + v) \tag{2}$$

where

$$A = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ -1 & 0 & 0 & \dots & 0 \\ 0 & -2 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & -2N-1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad z, B \in \mathbf{R}^{2N+2} \tag{3}$$

We use matrices

$$q_{ij} = \int_0^1 x^{i+j-2}(1-x)dx = \frac{1}{(i+j)(i+j-1)}, \quad i, j = 1, \dots, 2N+2$$
$$Q = q^{-1}, \quad C = -\frac{1}{2}B^T Q, \quad \delta(T) = \mathrm{diag}\{T^{-1}, T^{-2}, \dots, T^{-2N-2}\}$$

Note, that for the scalar control system (2),(3) the matrix C is a row-vector $C = (C_1, \dots, C_{2N+2})$. This allows to define the canonical feedback control by

$$u(z) = C\delta(T(z))z$$

which takes the form

$$u(z) = \frac{C_1}{T(z)}z_1 + \frac{C_2}{T^2(z)}z_2 + \dots + \frac{C_{2N+2}}{T^{2N+2}(z)}z_{2N+2} \tag{4}$$

in the scalar control case. The function $T(z)$ is found from the equation

$$\langle Q\delta(T)z, \delta(T)z \rangle = 1 \tag{5}$$

Theorem 2 *(see [2])*

A) Equation (5) define positive $T = T(z)$ uniquely.

B) The control (4) is bounded: $|u| \le \frac{1}{2}\sqrt{N(N+1)}$.

C) If there are no disturbances, the control (4) brings state z to 0 in time $T(z)$.

Theorem 3 *If disturbances v satisfy*

$$|v| \le c < \frac{1}{2\sqrt{N(N+1)}}$$

the derivative of T satisfies inequality

$$\dot{T} \le -\sigma, \ \sigma > 0$$

and control (4) brings state z to 0 in a finite time $T = O(T(z))$.

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References

[1] Brunovsky P. (1970) *Kibernetika* (6), 176-188.
[2] Ovseevich A. (2015) A Local Feedback Control Bringing a Linear System to Equilibrium *JOTA* 165, 532-544.