Influence of fractional viscoelastic connecting layers on the response of a beam-mass array exposed to motion of supports

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Summary. Herein the dynamic response of a system of fractional viscoelastic beams embedded in fractional viscoelastic medium and excited by motion of their supports is analysed, and the influence of properties of the connecting medium, modelled as a set of connecting layers, on the system behaviour is investigated. First, the approximate solution to the problem is obtained through the use of the Galerkin discretisation, impulse response method, Fourier transform and Residue theory, and then it is applied to analyse, both qualitatively and quantitatively, the influence of fractional-order derivative model parameters on the dynamic properties of beam arrays.

Introduction

There are many possible mechanical and engineering applications of systems of cantilever beams connected into an array and excited by motion of their supports, particularly for vibration attenuation and energy harvesting purposes [1]. One type of such systems, where cantilever beams are embedded in fractional viscoelastic medium, which is modelled by a set of light viscoelastic layers, and placed inside a moving container, is the subject of this study, and it is schematically depicted in Fig. 1. Dynamics of beam arrays with elastic and viscoelastic properties has already been investigated, e.g. [2]. There has also been some research regarding the fractional-order derivative viscoelastic systems (e.g. [3]), but these solutions are applicable only for rational derivative orders. Freundlich [4] recently presented the exact solution for the dynamic response of a single cantilever with a tip mass under transverse motion of the support, while the beam material was modelled with damping of an arbitrary order of fractional derivative. In our recent study, we have obtained an approximate solution to the problem of vibration of a system of fractional viscoelastic cantilevers, connected by a fractional viscoelastic layers and excited by transverse motion of the support, where the fractional derivatives of arbitrary order were used. In the herein presented paper, this solution procedure is briefly described, and then it is applied to analyse the previously mentioned array of connected cantilevers confined in a transversally moving container, schematically presented in Fig. 1, in order to determine the influence of the connecting layers' material properties on the dynamic response of the system.

Mathematical model and the approximate solution to the considered problem

A system of N_b fractional viscoelastic Euler-Bernoulli cantilever beams embedded in fractional viscoelastic medium and confined inside a transversally moving container, as presented in Fig. 1, is analysed. Each beam is of length L and carries $N_{m(k)}$ concentrated masses ${m_{(k)}}_p$ attached at $x_{m(k)} \in (0, L),\ p = 1, 2, \ldots, N_{m(k)},\ k = 1, 2, \ldots N_b$. Beams can have mutually different mass density ρ_k, cross sectional area A_k, moment of inertia I_k and relaxed elasticity modulus E_k, but they have the same fractional derivative order α and retardation time τ_1. The medium is modelled as a set of connecting layers, with prolonged compliance coefficient κ, fractional derivative order β and retardation time τ_2 being the same throughout the system. Here we will use only the left Riemann-Liouville fractional derivative of order γ as defined in [5], here denoted by $D^\gamma(\bullet) \equiv (\bullet)^{(\gamma)}$, $\gamma \in (0, 1)$. The container moves transversally, following an arbitrary function $w_s(t)$.

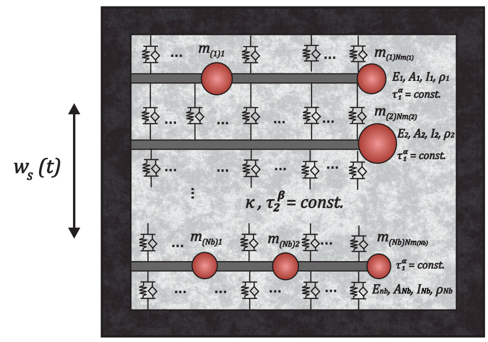

Figure 1: Schematic representation of the considered mechanical system

For the k-th beam of the system ($k = 1, 2, \ldots, N_b$), the equation of motion for transverse beam displacements $w_{(k)}(x, t)$ and the corresponding boundary conditions (BCs), noting that $w_{(0)} = w_{(N_b+1)} \equiv w_s$, can be formulated as

$$\begin{aligned} &E_k I_k (1 + \tau_1^\alpha D^\alpha) w_{(k)}'''' + \left(\rho_k A_k + \sum_{p=1}^{N_{m(k)}} m_{(k)p} \delta(x - a_{(k)p}) \right) \ddot{w}_{(k)} - \kappa (1 + \tau_2^\beta D^\beta)(w_{(k+1)} + w_{(k-1)}) + \\ &+ 2\kappa (1 + \tau_2^\beta D^\beta) w_{(k)} = 0 \, ; \qquad \text{BCs}: \quad w_{(k)}(0, t) = w_s(t), \quad w_{(k)}'(0, t) = w_{(k)}''(L, t) = w_{(k)}'''(L, t) = 0 \end{aligned} \tag{1}$$

The solution procedure and the influence of the connecting layer properties

In order to homogenise the BCs, the absolute displacements are decomposed into the rigid body motion part and the displacements relative to the supported beam end - $w_{(k)}(x,t) = w_s(t) + v_{(k)}(x,t)$. Then, the relative displacements are approximated by the Galerkin weighted residual method as $v_{(k)}(x,t) \approx \sum_{i=1}^{n} \phi_{(k)i}(x) q_{(k)i}(t)\,, \quad k = 1, 2, \ldots, N_b$, with the bare beam mode shapes used as the trial functions and the weighting functions. This leads to a system $n \times N_b$ coupled fractional-order differential equations which can be expressed in matrix form as

$$\mathbf{Kq} + \mathbf{C}_\alpha \mathbf{q}^{(\alpha)} + \mathbf{C}_\beta \mathbf{q}^{(\beta)} + \mathbf{M}\ddot{\mathbf{q}} = \mathbf{Q} \tag{2}$$

where $\mathbf{K}$, $\mathbf{C}_\alpha$, $\mathbf{C}_\beta$ and $\mathbf{M}$ are the stiffness matrix, beam material damping matrix, connecting layer damping matrix and mass matrix of the whole system, respectively, and $\mathbf{q}$ and $\mathbf{Q}$ are the vector of the yet undetermined time functions and the vector of the inertial forces in the whole system. The unknown time functions $\mathbf{q}^{\mathrm{T}} = \{\mathbf{q}_1, \mathbf{q}_1, \ldots, \mathbf{q}_{N_b}\} = \{{q_{(1)}}_1, {q_{(1)}}_2, \ldots, {q_{(1)}}_n, {q_{(2)}}_1, \ldots, {q_{(N)}}_n\}$ will be determined by first finding the system impulse response $\mathbf{g}$, where $\mathbf{g}$ is the vector of $n \times N_b$ corresponding Green functions $G_i(t),\ i = 1, 2, \ldots, n \times N_b$.

The impulse response is determined by taking the Fourier transform of the system and then using the equivalent elastic system to obtain the decoupled system of equations, assuming relatively small damping. This leads to (almost) diagonalised system matrices, i.e. a system of $n \times N_b$ decoupled polynomial algebraic equations with fractional exponents:

$$s_r^2 + C_{\alpha rr}^d s_r^\alpha + C_{\beta rr}^d s_r^\beta + \omega_r^2 = 0\,, \qquad r = 1, 2, \ldots, n \times N_b \tag{3}$$

where $s = \imath\omega$, with $\omega_r^2 = K_{rr}^d / M_{rr}^d$ being the r-th undamped system frequency, and $C_{\alpha rr}^d$, $C_{\beta rr}^d$, K_{rr}^d, M_{rr}^d are the r-th diagonal elements of the corresponding diagonalised matrices. After finding the roots of each of these equations, the inverse Fourier transform is applied with the use of the Residue theory, and the Green functions are again coupled to obtain the impulse response of the original system. Once the impulse response is determined, the sought time functions are obtained by taking the convolution with the inertial forces of the system $q_i(t) = \int_0^t G_i(t-\tau) Q_i(\tau) d\tau\,,\ i = 1, 2, \ldots, n \times N_b$, thus providing the complete (approximate) solution to the considered problem. The influence of the connecting layer material properties was investigated on a system of $N_b = 3$ cantilever beams with $N_{m(1)} = 2$, $N_{m(2)} = 1$, $N_{m(3)} = 3$ equidistantly attached masses of half of each beam's weight. Geometrical and material properties of each beam were adopted the same as in [4], with $\tau_1 = 0.001s,\ \alpha = 0.8$. The container was set to follow the motion function $w_s(t) = w_0 \sin \omega_s t^2$, with $w_0 = 1mm,\ \omega_s = 10s^{-1}$. The parameters τ_2, β and κ were varied, and their influence on the relative displacements of the free end of the first beam in the array are presented in Fig. 2.

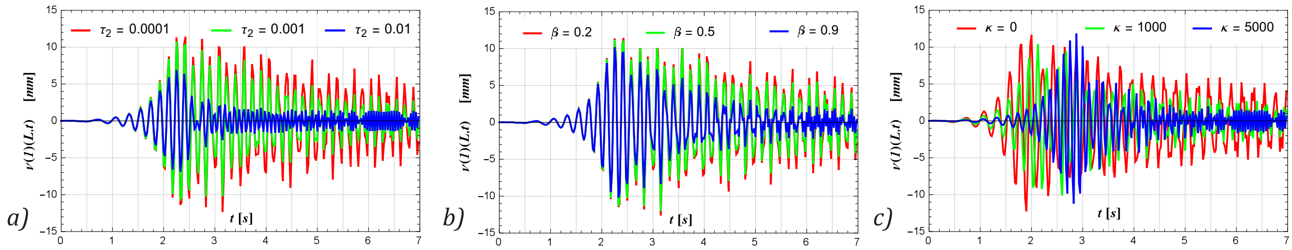

Figure 2: Influence of the connecting layer properties on the dynamic response of the system

Conclusions

It can be seen that the connecting layer greatly affects system behaviour. An increase in retardation time τ_2 and order of fractional derivative β of the connecting layer leads to a more rapid vibration attenuation, as shown in Fig. 2a) and b), respectively, while an increase in the layer's relaxed compliance coefficient leads not only to an increase in damping, but in the fundamental system frequency as well, which causes the resonant state shift observed in Fig. 2c).

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