Differential geometric PDE bifurcation problems in `pde2path`

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Summary. Shape equations, such as the constant mean curvature (CMC) equation or the Helfrich equation, are geometric partial differential equations that arise for example in modeling liquid bridges or red blood cells. Analytical solution of these equations are known only in few situations, and we use numerical continuation to compute solutions and study their parameter dependence. The problems require adaptations of typical PDE numerics because of the quasilinear nature of the equations and the geometrical background. We discuss how some of these issues are treated in the numerical continuation and bifurcation package `pde2path`. To illustrate the implementation, we present some examples of CMC and Helfrich surfaces.

Introduction

Shape equations are differential geometric partial differential equations that typically describe critical points of some energy functional under some constraints. For example, let M be a two dimensional surface (possibly with boundary) immersed in $\mathbb{R}^3$ and parametrized by $\varphi : \Omega \to M$. Then the area functional is

$$\text{area(M)} = \int_M \mathrm{d}S. \tag{1}$$

Now adding a volume constraint one finds that constant mean curvature (CMC) surfaces are critical points, leading to the following boundary value problem: given $V_0 \in \mathbb{R}_+$, find $\varphi : \Omega \to \mathbb{R}^3$ and $H_0 \in \mathbb{R}$ such that

$$H(\varphi) - H_0 = 0, \quad \text{Vol}(\varphi) - V_0 = 0, \quad \varphi|_{\partial\Omega} = \partial M, \tag{2}$$

where H is the mean curvature, $V(\varphi)$ the enclosed volume, and the constant H_0 can for instance be interpretated as a pressure difference when the CMC surfaces model interfaces between fluids.
In general, only a few solutions of (2) and of related (often more complicated) problems are known explicitly, and we have to resort to numerical approximation of solutions. There are various methods for this, see, e.g., [BGN20] for a review of sophisticated methods. Here we follow an approach from [Bru18] and aim to numerically continue solutions of (2) and related problems in parameters, within the framework of the `Matlab` continuation and bifurcation package `pde2path` [Uec21a, Uec21b]. The basic idea is as follows: Let M_0 be a known surface with parametrization $\varphi_0 : \Omega \to \mathbb{R}^3$ satisfying (2) for some V_0 and with mean curvature H_0, and define a new surface via the parametrization $\varphi = \varphi_0 + u\,\nu$, $u : \Omega \to \mathbb{R}$ with suitable boundary conditions, where $\nu : M_0 \to \mathbb{S}^2$ is the unit normal vector of M_0. Then (2) reads

$$F(u, H, V) = \begin{pmatrix} H(\phi) - H \\ V(\phi) - V \end{pmatrix} = 0, \text{ with boundary conditions for } u, \tag{3}$$

which is a quasilinear elliptic equation for u, and after solving (3) we can update $M_0 = M_0 + u\nu$ and repeat. This point of view is also useful analytically, see, e.g., [ES98], and numerically it allows to apply standard predictor–corrector continuation and bifurcation methods to the quasilinear elliptic system (3). In `pde2path`, the (default) discretization of (3) works by the finite element method (FEM), and the associated mesh adaption capabilities turn out to be very useful to deal with possibly strong deformations of the manifolds M under the continuation.
The method can also be applied to other types of geometric PDEs, also of higher order, for instance fourth order models from mathematical biology, e.g., Willmore and Helfrich functionals for vesicle shapes. In this case, (3) can be rewritten as a system of (2nd order) PDEs for a vector valued (u_1, u_2), and the same ideas apply.

Examples

Simple examples of CMC surfaces with boundaries are spherical caps, for instance the one–parameter CMC family of spherical caps S immersed in $\mathbb{R}^3$ with the boundary $\partial M = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \le 1, z = 0\}$. These are useful testing problems since a spherical cap with prescribed boundary is uniquely characterized by its volume, and we have an analytical relation between volume and mean curvature, which yields an explicit error analysis. Figure 1 shows a simple continuation of the caps in volume, starting from $H = 0$ (flat disk), and applying some mesh adaptation when under continuation the stretching of the triangles becomes large.
For the spherical caps it is known and intuitively clear that no bifurcations occur along the branch in Fig. 1, cf. [KPP15]. This is different for another classical example, namely liquid bridges [SAR97]. In Fig. 2 we consider CMC surfaces between two unit circles located at $z = \pm 1/2$. For $V = \pi$ we have a straight cylinder with $H_0 = 1/2$, and continuing to larger V we find bifurcations to non–axisymmetric surfaces with 1, 2, ... bulges, with the first bifurcation occuring at $V \approx 5.81$ where the surface meets the planes $z = \pm 1/2$ at the circles tangentially. Here, for efficiency we compute only half the bridges with Neumann boundary conditions in angle, and the algorithm works fast and robustly (including mesh adaptation).

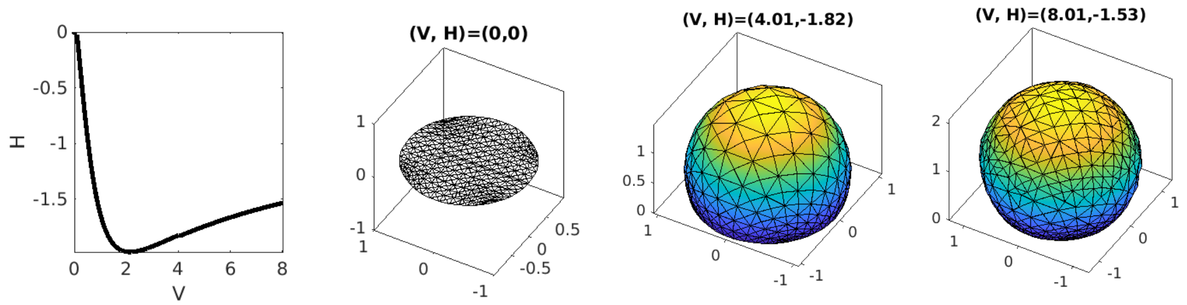

Figure 1: $2H$ over V for the spherical cap example, and sample solutions. At $V = 4.01$ and $V = 8.01$, the colors indicate u in the last step; at $V = 4.01$ we adaptively refine the mesh, see $V = 8.01$.

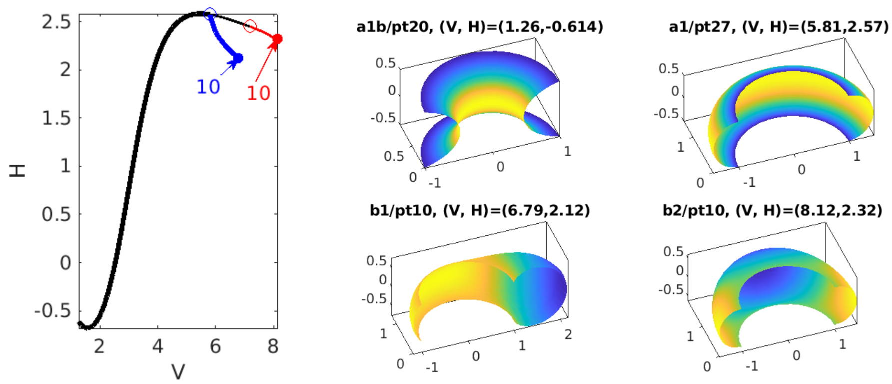

Figure 2: Bifurcation diagram of liquid bridges, and sample solutions. Top row: axisymmetric (black branch), with pt27 near the 1st BP. Bottom: pt10 on blue (left) and red (right) branches. The colors again indicate the last u.

After the above standard examples (also considered in [Bru18]), we consider more complicated problems, including: liquid bridges in different geometries, under gravity and with further terms, bifurcating nodoids [MP02, KPP15] and triply periodic surfaces [KPS18], and time permitting give an outlook on bifurcation results for 4th order equations such as Canham–Helfrich equations for vesicles modeling for instance red blood cells [Sei97].

References

[BGN20] J. Barrett, H. Garcke, and R. Nürnberg. Parametric finite element approximations of curvature-driven interface evolutions. In *Geometric partial differential equations. Part I*, volume 21 of *Handb. Numer. Anal.*, pages 275–423. Elsevier/North-Holland, Amsterdam, [2020] ©2020.

[Bru18] N. D. Brubaker. A continuation method for computing constant mean curvature surfaces with boundary. *SIAM J. Sci. Comput.*, 40(4):A2568–A2583, 2018.

[ES98] J. Escher and G. Simonett. The volume preserving mean curvature flow near spheres. *Proc. Amer. Math. Soc.*, 126(9):2789–2796, 1998.

[KPP15] M. Koiso, B. Palmer, and P. Piccione. Bifurcation and symmetry breaking of nodoids with fixed boundary. *Adv. Calc. Var.*, 8(4):337–370, 2015.

[KPS18] M. Koiso, P. Piccione, and T. Shoda. On bifurcation and local rigidity of triply periodic minimal surfaces in $\mathbb{R}^3$. *Ann. Inst. Fourier (Grenoble)*, 68(6):2743–2778, 2018.

[MP02] R. Mazzeo and Fr. Pacard. Bifurcating nodoids. In *Topology and geometry: commemorating SISTAG*, volume 314 of *Contemp. Math.*, pages 169–186. Amer. Math. Soc., Providence, RI, 2002.

[SAR97] L. Slobozhanin, J. Alexander, and A. Resnick. Bifurcation of the equilibrium states of a weightless liquid bridge. *Phys. Fluids*, 9(7):1893–1905, 1997.

[Sei97] U. Seifert. Configurations of fluid membranes and vesicles. *Advances in Physics*, 46(1):13–137, 1997.

[Uec21a] H. Uecker. *Numerical continuation and bifurcation in Nonlinear PDEs*. SIAM, Philadelphia, PA, 2021.

[Uec21b] H. Uecker. `www.staff.uni-oldenburg.de/hannes.uecker/pde2path`, 2021.