A surface of connecting orbits between two saddle slow manifolds in a return mechanism of mixed-mode oscillations

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Summary. We employ a Lin's method set-up to compute a surface of heteroclinic connections between two saddle slow manifolds in the four-dimensional Olsen model for peroxidase-oxidase reaction. As will be shown, this surface organises the return mechanism of mixed-mode oscillations that involve a slow passage through a Hopf bifurcation.

We consider a model for peroxidase-oxidase reaction first introduced by Olsen [6], which we study here in the scaled form presented in [4]; it is given as the system of ordinary differential equations

$$\left\{\begin{array}{rcl} \dot{A} &=& \mu - \alpha A - ABY, \\ \dot{B} &=& \varepsilon(1 - BX - ABY), \\ \dot{X} &=& \lambda(BX - X^2 + 3ABY - \zeta X + \delta), \\ \dot{Y} &=& \kappa\lambda(X^2 - Y - ABY), \end{array}\right. \tag{1}$$

for the vector of chemical concentrations $(A, B, X, Y) \in \mathbb{R}^4$. The system parameters are fixed here, as in [4], to $\alpha = 0.0912$, $\delta = 1.2121 \times 10^{-4}$, $\varepsilon = 0.0037$, $\lambda = 18.5281$, $\kappa = 3.7963$, $\mu = 0.9697$, and $\zeta = 0.9847$. For this choice, the three concentrations A, X, and Y can be considered as fast and B as slow. System (1) has been of interest because it exhibits mixed-mode oscillations (MMOs), which are characterised by a mixture of small-amplitude oscillations (SAOs) that usually arise locally in phase space and large-amplitude oscillations (LAOs) that are generally associated with a global return to the region of SAOs. To date, mechanisms for MMOs are quite well understood in slow-fast systems of dimension three; see, for example, the survey paper [5] as an entry point to the literature on MMOs. For four-dimensional slow-fast systems, on the other hand, new mechanisms and types of MMOs may arise. The case study of the Olsen model (1) presented here shows that the return mechanism of the MMOs involves heteroclinic connecting orbits between two saddle slow manifolds. It follows on from earlier work in [1], where the same parameter regime was considered but a model reduction to a three-dimensional system (via a quasi-steady-state assumption) was performed. In contrast, we now consider and compute all relevant objects in the full (A, B, X, Y)-space of system (1).

In the spirit of geometric singular perturbation theory [3], we start with the limit of $\varepsilon = 0$ and consider the three-dimensional fast subsystem for the fast variables A, X, and Y, where the slow variable B is now a parameter. The equilibria of the fast subsystem, which are parametrised by B, form the critical manifold C in the (A, B, X, Y)-space of system (1). A linear stability analysis shows that C consists (in the physically relevant region of positive A, B, X, and Y) of four branches of hyperbolic equilibria of the fast subsystem: a branch denoted C^4_+ of stable equilibria; a branch C^3 of saddle equilibria with one unstable eigenvalue; a branch C^2 of saddle equilibria with two unstable eigenvalues; and a second branch C^4_- of stable equilibria. These branches connect at points F_1 and F_2 of fold bifurcation and H of Hopf bifurcation, and the superscipts indicate the dimensions of their stable manifolds in (A, B, X, Y)-space.

Our specific interest is in the two branches C^3 and C^2 because they are saddle objects with different dimensions of stable and unstable manifolds. While C^3 and C^2 only exist for $\varepsilon = 0$, according to Fenichel threory [3], they persist as locally invariant slow manifolds S^3 and S^2 for sufficiently small $\varepsilon > 0$; note that orbit segments on a slow manifold remain slow for $O(1)$ time. Moreover, the one-dimensional manifolds S^3 and S^2 lie $O(\varepsilon)$ close to C^3 and C^2, and they have stable and unstable manifolds of the same dimensions as those of C^3 and C^2. Hence, S^3 has a three-dimensional stable manifold $W^s(S^3)$, consisting of orbit segments that, in forward time, approach S^3 along a fast direction and then remain slow while following S^3; similarly, S^2 has a three-dimensional unstable manifold $W^u(S^2)$ consisting of orbit segments that, in backward time, approach S^3 along a fast direction and then remain slow while following S^3.

The two three-dimensional objects $W^s(S^3)$ and $W^u(S^2)$ are expected to intersect generically in a two-dimensional surface $\mathcal{H}$ of connecting orbits between S^3 and S^2; in forward time, any such connecting orbit first slowly follows the curve S^2, makes a transition across to the curve S^3, and then follows it slowly. In order to find $\mathcal{H}$, we use two ingredients: firstly, we adapt the technique in [2] for the computation of one-dimensional slow manifolds and their (un)stable manifolds to the four-dimensional setting of system (1) and, secondly, we employ a Lin's method approach [7] to define two orbit segments, in $W^s(S^3)$ and $W^u(S^2)$, respectively, that have end points in a chosen three-dimensional section. Closing the gap between them along a specified direction, by continuation of solutions to an overall boundary value problem, allows us to find a first heteroclinic orbit, which is then swept out in a further continuation run to obtain the surface $\mathcal{H}$.

Figure 1 shows the two-dimensional surface $\mathcal{H}$ in projection onto the three-dimensional (B, A, X)-space of system (1), together with the critical branches C^4_+ , C^2, C^4_- and C^3. Notice that $\mathcal{H}$ spirals out from the saddle branch C^2 and then approaches the saddle branch C^3 in a non-spiralling fashion; here, S^2 and S^3 (not shown) are indistinguishably close to

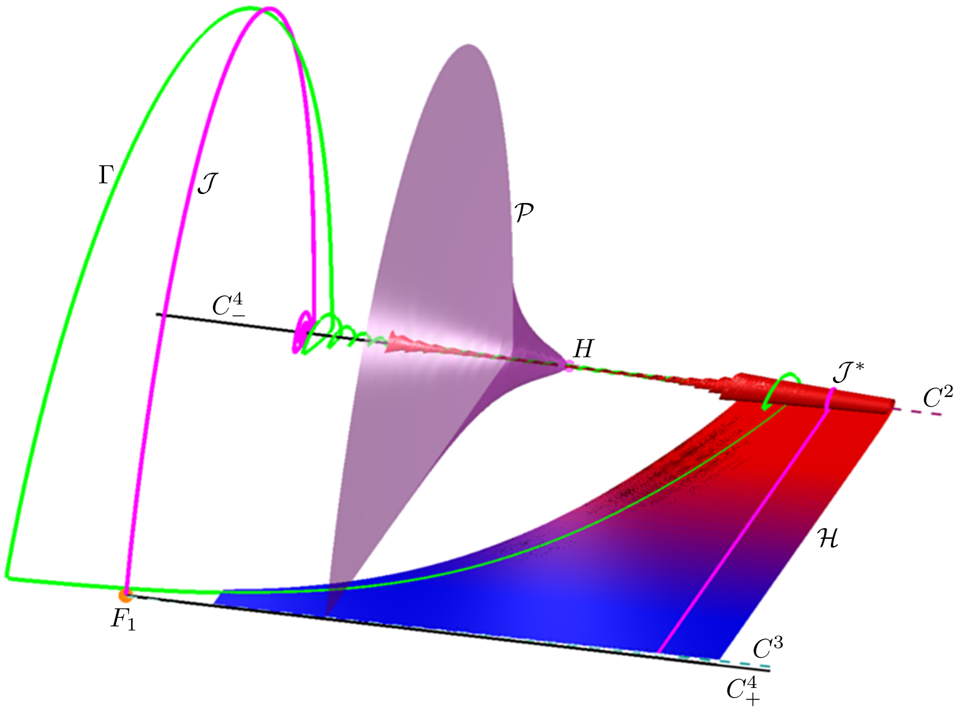

Figure 1: Three-dimensional projection onto (B, A, X)-space of the MMO periodic orbit Γ and the surface $\mathcal{H} = W^s(S^3) \cap W^u(S^2)$ (red-blue faded) of system (1) for $\varepsilon = 0.0037$, shown in relation to the following objects for $\varepsilon = 0$: the curves C^4_+ (black), C^2 (dashed raspberry), C^4_- (black) and C^3 (dashed teal) of the critical manifold; the fold point F_1 (orange dot) and the Hopf bifurcation point H (pink dot); the singular jump branch $\mathcal{J}$ from F_1 to C^4_- and its counterpart $\mathcal{J}^*$ (magenta curves) from the counterpoint on C^2, at equal distance from H, to C^3; and the surface $\mathcal{P}$ (transparent midnight grape) of periodic orbits arising from H.

C^2 and C^3 on the scale of Fig. 1. The surface $\mathcal{H}$ consists of orbit segments of an intermediate timescale in both $W^s(S^3)$ and $W^u(S^2)$, namely those that "make it all the way across" in forward time from near C^2 (where $\mathcal{H}$ is shaded red) to near C^3 (where $\mathcal{H}$ is shaded blue). As we checked, orbit segments close to but not on $\mathcal{H}$ quickly diverge from this surface in both forward and backward time in the X- and Y-directions.

Figure 1 is for $\varepsilon = 0.0037$ when one finds a stable MMO periodic orbit Γ, which is also shown. Starting near the attracting branch C^4_-, observe the SAOs of decreasing and then increasing amplitude that are generated by a slow passage through the Hopf bifurcation point H. Well past H and the surface $\mathcal{P}$ of periodic orbits of the fast subsystem, Γ leaves the branch C^2 by following the surface $\mathcal{H}$ very closely to a vicinity of the critical branch C^3. Somewhat past the fold point F_1, it subsequently has a sudden excursion in the X- and Y-directions to return back to C^4_-; the process then repeats.

We conclude that the surface $\mathcal{H}$ of connecting orbits is a crucial part of the return mechanism that is responsible for a single LAO per period of Γ. We further observe that Γ returns to C^4_- very near where the critical jump orbit $\mathcal{J}$ from F_1 for $\varepsilon = 0$ returns (this point is given by the B-value of F_1). Similarly, the take-off point on C^2 is close to that of the counterpart $\mathcal{J}^*$ of $\mathcal{J}$, which lies on the other side of $\mathcal{H}$ at the same B-distance. The suggestion from Fig. 1 is, hence, that the concatenation of $\mathcal{J}$ and $\mathcal{J}^*$ with the respective parts of C^4_-, C^2 and C^3 acts as a singular limit of Γ as ε approaches 0. Indeed, how the MMOs of the Olsen model (1) depend on parameters is a subject of our ongoing research.

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