Asymptotic formulation of bifurcation scenarios to post-buckling nonlinear vibrations in thermomechanically coupled plates

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Summary. The nonlinear dynamics of composite plates with thermomechanical coupling is analytically addressed in order to describe the main dynamical phenomena triggering the involved pre- and post-buckling response scenario. The statical buckling occurrence, and two resonance conditions around the unbuckled and buckled equilibria are investigated by means of the asymptotic multiple scale method. The resulting modulation equations and the steady state mechanical and thermal responses are determined and compared with the numerical outcomes in order to verify the effectiveness of the adopted procedures.

Introduction

The nonlinear dynamical behavior of reduced order models of composite plates under different excitation conditions in a thermomechanical environment has been the subject of recent papers aimed at highlighting the role of multiphysics coupling and the main local and global features of the nonlinear response [1, 2]. The numerical analyses, carried out in strongly nonlinear regime and under different mechanical and thermal conditions, highlighted a rich and involved scenario characterized by multistability and possible chaos. Yet, the analytical treatment of the dynamical response can represent a useful tool to predict, describe and possibly modify the behavior of the coupled system. To this aim, three main response phenomena are detected in the weakly nonlinear regime, and the asymptotic multiple scale method [3] is applied in order to investigate existence and stability of the mechanical and thermal responses of the system.

Asymptotic analysis

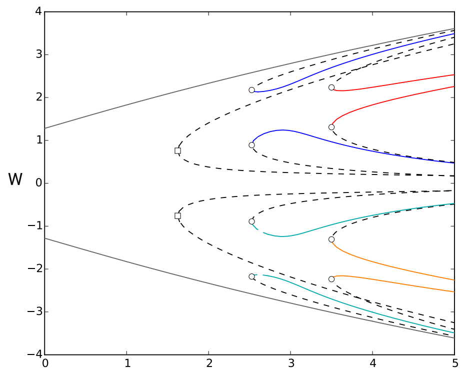
With reference to a reduced model of rectangular laminated plate with von Kármán nonlinearities, third-order shear deformability and consistent cubic variation of the temperature along the thickness [4], the assumption of isothermal edges and free heat exchange on the upper and lower surfaces leads to obtain the following nondimensional equations of motion describing the plate dynamics around primary resonance:

$$\begin{aligned}
&\ddot{W}(t) + a_{12}\dot{W}(t) + (\Omega^2 - p)W(t) + a_{14}W(t)^3 + a_{15}T_{R1}(t) + a_{16}T_{R0}(t)W(t) - f\cos\Omega t = 0\\
&\dot{T}_{R0}(t) + a_{22}T_{R0}(t) + a_{23}\alpha_1 T_\infty + a_{24}W(t)\dot{W}(t) = 0\\
&\dot{T}_{R1}(t) + a_{32}T_{R0}(t) + a_{33}\dot{W}(t) = 0
\end{aligned} \tag{1}$$

in terms of the nondimensional reduced variables $W(t)$ (deflection of the center of the plate), $T_{R0}(t)$ (membrane temperature) and $T_{R1}(t)$ (bending temperature). The mechanical excitations consist of a uniform compressive force p on the plate edges and a distributed harmonic transverse mechanical excitation of amplitude f and frequency Ω. The thermal excitation is represented by the constant thermal difference between plate and environment T_∞, while a_{ij} are coefficients which incorporate the geometrical and physical properties of the model.

Local and global nonlinear dynamics of the presented model have been investigated by parametrically accounting for the single and combined presence of thermal and mechanical excitations. In particular, the transition to mechanically- or thermally-induced buckled responses has been analyzed, and a variety of rich multistable scenarios have been detected, as exemplarily shown in terms of numerical bifurcation diagrams as a function of the mechanical pretension in Fig. 1(a). With the aim to unveil the bifurcation phenomena triggering the development of such a rich scenario, identified in a strongly non-linear regime, a lower harmonic forcing amplitude has been applied to the system, with the relevant bifurcation diagram reported in Fig. 1(b). The results allow us to detect three main underlying dynamical features, i.e., a static pitchfork bifurcation inducing the mechanical buckling, and two resonance peaks occurring in the pre- and post-buckling branches. The first phenomenon occurs when the mechanical pretension p nullifies the linear mechanical stiffness, while the two peaks correspond to primary resonances of the pre- and post-buckling system frequencies with the external harmonic excitation.

The static buckling analysis is performed by obtaining the equilibria $\boldsymbol{e} = \{\boldsymbol{W_e}, \boldsymbol{T_{R0e}}, \boldsymbol{T_{R1e}}\}$ of the coupled system (1), which have the following expressions:


(a)

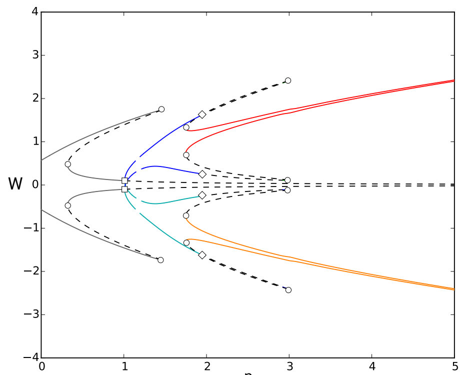

(b)

Figure 1: Bifurcation diagrams, with detection of the maximum and minimum values of the mechanical response as a function of p, at $\Omega = 1$, for $f = 1$ (a) and $f = 0.1$ (b).

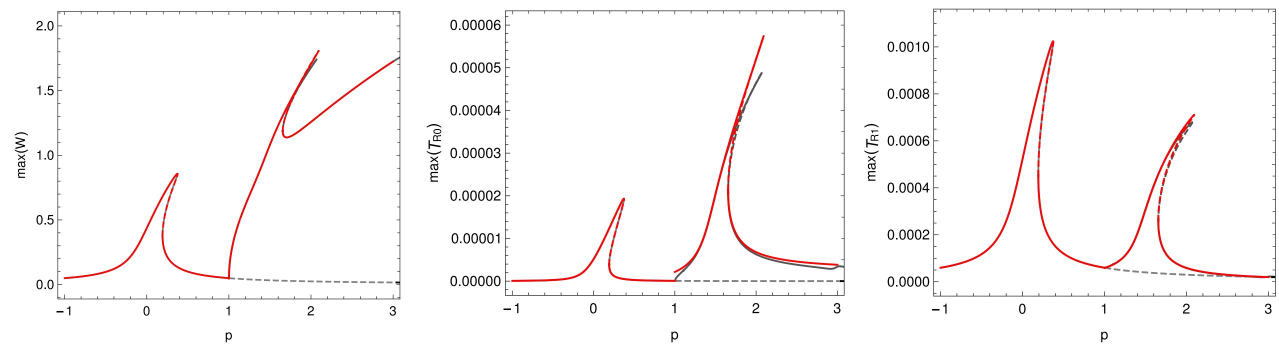


Figure 2: Comparison between numerical (gray) and analytical (red) bifurcation diagrams for $\Omega = 1$ and $f = 0.05$.

$$\boldsymbol{e_1} = \{0, -\frac{a_{23}\alpha_1 T_\infty}{a_{22}}, 0\}, \qquad \boldsymbol{e_{2,3}} = \{\pm\frac{\sqrt{a_{22}(p-\Omega^2)+a_{16}a_{23}\alpha_1 T_\infty}}{\sqrt{a_{14}a_{22}}}, -\frac{a_{23}\alpha_1 T_\infty}{a_{22}}, 0\} \tag{2}$$

The $\boldsymbol{e_1}$ equilibrium corresponds to the pre-buckling configuration representing the mechanical rest position, while $\boldsymbol{e_2}$ and $\boldsymbol{e_3}$ represent the two stable buckled non-trivial solutions arising after the pitchfork bifurcation. As already highlighted in some previous works [1, 2], expressions (2) show that the thermal excitation T_∞ plays the same role as the mechanical pretension in governing the mechanical equilibria, so that it is possible to reproduce exactly the diagrams of Fig. 1 by alternatively applying a properly scaled thermal excitation. The two resonance conditions are analytically investigated by means of the asymptotic method of multiple scales, in order to study the system nonlinear dynamics around the previously obtained $\boldsymbol{e_1}$ and $\boldsymbol{e_2}$ equilibria: $W(t) = W_e + \tilde{W}$, $T_{R0}(t) = T_{R0e} + \tilde{T}_{R0}$, $T_{R1}(t) = T_{R1e} + \tilde{T}_{R1}$. The perturbation equations around the pre-buckling equilibrium have the same structure of Eqs. (1) (with suppression of a_{23} term in the membrane thermal equation), with the system frequency (i.e., the time-independent linear stiffness) being $\omega^2 = \Omega^2 - p - a_{16}a_{23}\alpha_1 T_\infty/a_{22}$. Conversely, the analysis around the buckled position implies the presence of additional terms in the mechanical and membrane thermal equations:

$$\begin{aligned}
&\ddot{\tilde{W}} + a_{12}\dot{\tilde{W}} + \omega^2\tilde{W} + a_{14}\tilde{W}^3 + \frac{3\sqrt{a_{14}}}{\sqrt{2}}\omega\tilde{W}^2 + a_{15}\tilde{T}_{R1} + a_{16}(\frac{\omega}{\sqrt{2a_{14}}} + \tilde{W})\tilde{T}_{R0} - f\cos(\Omega t) = 0\\
&\dot{\tilde{T}}_{R0} + a_{22}\tilde{T}_{R0} + a_{24}\dot{\tilde{W}}(\frac{\omega}{\sqrt{2a_{14}}} + \tilde{W}) = 0\\
&\dot{\tilde{T}}_{R1} + a_{32}\tilde{T}_{R0} + a_{33}\dot{\tilde{W}} = 0
\end{aligned} \tag{3}$$

with $\omega^2 = 2(p + a_{16}a_{23}\alpha_1 T_\infty/a_{22} - \Omega^2)$. Consequently, the two asymptotic procedures require different choices in the scaling of variables and parameters, and different expansions to higher orders to account for the main effects (e.g., the occurrence of both quadratic and cubic nonlinearities, as in (3)). In both cases, anyway, they have been guided by the previous numerical analyses, which have pointed out the contemporary presence of slow (thermal) and fast (mechanical) dynamics, and have allowed to discuss the role of the coupling terms inside the three equations [5]. They result to be crucial into the thermal equations in order to determine the temperature response, while having a marginal effect on the mechanical equation, whose dynamics evolves much quicker than the coupled thermal one. The two procedures are developed separately, and the Amplitude Modulation Equations are obtained together with the reconstructed steady state mechanical and thermal responses. The outcomes reported in Fig. 2 show a good agreement between analytical (red) and numerical (gray) results, also in the post-buckling scenario where the mechanical response is moderately severe.

Conclusions

The analytical treatment through the multiple scale method is developed to describe the main dynamical phenomena underlying the rich multistable scenario characterizing the nonlinear behavior of thermomechanically coupled plates. The obtained modulation equations together with the reconstructed responses can be used to parametrically discuss the occurrence and stability of the main periodic unbuckled and buckled responses as a function of the main system parameters.

References

[1] Settimi V., Rega G., Saetta E. (2018) Avoiding/inducing dynamic buckling in a thermomechanically coupled plate: a local and global analysis of slow/fast response. *Proc. R. Soc. Lond., A*, **474(2213)**:20180206.

[2] Settimi V., Saetta E., Rega G. (2019) Nonlinear dynamics of a third-order reduced model of thermomechanically coupled plate under different thermal excitations. *Meccanica*, to appear.

[3] Nayfeh A.H., Mook D.T. (1979) Nonlinear oscillations. Wiley, NY.

[4] Saetta E., Rega G. (2017) Third-order thermomechanically coupled laminated plates: 2D nonlinear modeling, minimal reduction and transient/post-buckled dynamics under different thermal excitations. *Compos. Struct.*, **174**:420-441.

[5] Settimi V., Rega G. (2019) Thermomechanical coupling and transient to steady global dynamics of orthotropic plates. *in* Problems of Nonlinear Mechanics and Physics of Materials, Springer, pp:483-499.