Numerical Methods for Nonsmooth DAEs

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Summary. In this work, we will discuss the various approaches to simulate hybrid differential algebraic equations (hybrid DAEs), i.e., dynamical systems with some algebraic constraints switching with respect to the state variables. Then, we will present our recent work on the simulation of such hybrid DAE through a reformulation as DAEs with non-smooth constraints. Finally, we show on some examples that numerical methods for non-smooth dynamical systems can be extended for the simulation of these non-smooth DAEs.

Hybrid DAEs are an increasing share of the simulated hybrid systems. They can be found in a large panel of engineering domains such as, but not limited to, chemical process engineering [15], electronics [1], for a long time in mechanical systems [4], and in numerous cyber-physical systems. In most of these domains, engineers rely on model-based design language such as Modelica [8] to define their dynamics in a piecewise manner using conditional statements. The resulting dynamical systems are of the form:

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{z}(t)) \\ \quad 0 = \mathbf{g}_i(\mathbf{x}(t), \mathbf{z}(t)) \\ \qquad , \forall (\mathbf{x}(t), \mathbf{z}(t)) \in \mathcal{X}_i. \end{cases} \tag{1}$$

The variables $\mathbf{x}, \mathbf{z}$ are the differential and algebraic variables, respectively. The sets $\mathcal{X}_i = \{(\mathbf{x}, \mathbf{z}) \in \mathbb{R}^{n_1+n_2=n} \mid \mathbf{h}_i(\mathbf{x}, \mathbf{z}) > 0\}$ define a partition of $\mathbb{R}^n$ such that: $\bigcup_i \overline{\mathcal{X}}_i = \mathbb{R}^n$, $\mathrm{int}(\mathcal{X}_i) \neq \emptyset$, for all i. In addition, let assume that $i \neq j$, $\partial\mathcal{X}_i \cap \partial\mathcal{X}_j = \emptyset$. Indeed, the definitions of the dynamics on the border of a partition are, in general, ambiguous.

Related Works

It is first worth to mention that some works, such as [14], have taken the approach of relaxing the non-smoothness inherent to the hybrid DAE using smooth versions of a step function such as the sigmoid function, Hill function, or the hyperbolic tangent. If these works enable a direct application of known numerical methods for non-linear DAEs, they, in general, worsen the numerical simulation efficiency due to the high stiffness, but also the numerical stability because of bad conditioning inherent to stiff systems. This was for example noted for the simulation of electronic systems with diodes [1].

Another approach, taken by V. Merhmann et al. [7, 11], is to use numerical methods defined for hybrid DAE structured as hybrid automata. Their implementation relies on event based numerical methods and explicit transition functions to correctly re-initialise the system after switching. It is noteworthy that they detect and simulate in some particular case sliding modes in a similar fashion to Filipov sliding motion [6] for ODEs. In particular, this concept of continuous solutions for hybrid DAE as been formalised and studied by I.V Matrosov [9, 10] in an unrelated work. The work of A. Benveniste et al. [3] uses non-standard analysis to construct well-defined transitions from one mode to another in the context of hybrid DAE even in presence of varying index. In particular, this work pairs well with [11] as the definition by the user of the DAE initialisation at switching is not needed anymore. It is also worth to mention the work of S. Trenn [16] that defines the solutions of hybrid DAE with exogenous switching. In particular, he introduces the notion of distributional solutions which can also be used to efficiently solve inconsistent initial conditions.

Outside of the hybrid automata approach, the hybrid DAE have also been studied as non-smooth dynamical systems. For example, differential variational inequalities (DVI) form a class of non-smooth DAEs which are studied in [12]. In particular, they give well-posedness results, and some numerical methods are analysed, for DVI with a structure similar to index-1 DAEs. K. Camlibel et al. [5] extend results of well-posedness of differential inclusions to differential algebraic inclusions with maximal monotone operators. In additions, they study the well-posedness of non-smooth linear DAEs with some passivity properties.

Analysis of Non-smooth DAEs

In a previous work [13], we proposed to relax the switching algebraic equations $\mathbf{g}_i(\mathbf{x}, \mathbf{z}) = 0$ from (1) by filling-in the graph of the constraints. This is achieved by using step-functions $\mathrm{s}^+(\cdot)$ in a similar fashion to [2] in order to build a generalised constraint.

$$\mathbf{g}(\mathbf{x}, \mathbf{z}) = \sum_i \left(\prod_{j \neq i} (1 - \mathrm{s}^+(\mathbf{h}_j(\mathbf{x}, \mathbf{z}))) \right) \mathrm{s}^+(\mathbf{h}_i(\mathbf{x}, \mathbf{z})) \mathbf{g}_i(\mathbf{x}, \mathbf{z}) = 0\,. \tag{2}$$

The actual definition of $\mathbf{g}(\mathbf{x}, \mathbf{z})$ will depend of the choice of the step-function definition when $h_j(\mathbf{x}, \mathbf{z}) = 0$. Then, we studied the effect of such relaxation in (3), a particular 2-dimensional example (see Figure 1), and we have shown that the constraint (2) is relaxed as a generalised equation, whose well-posedness can be studied. We show that the system with the generalised constraint (2) presents continuous sliding solutions that are not exhibited by the methods of [11] and [9].

$$\begin{cases} \dot{x}_1(t) = 1 + \mathrm{B}_1 z(t) \\ \dot{x}_2(t) = \mathrm{B}_2 z(t) \\ \qquad 0 \in \operatorname{sign}(x_1(t)) + |x_1(t))| - x_2(t)\,. \end{cases} \tag{3}$$

Then, by studying the well-posedness of the implicit Euler numerical discretization of this example (4), we conjecture an extension of the implicit Euler numerical scheme for simulation of such non-smooth DAEs. This improved Euler numerical scheme has proven to yield good results on our example (see convergence results on Fig. 2).

$$\begin{aligned} p^*_{k+1} := \min_{\mathbf{x}_{k+1},\mathbf{z}_{k+1},\boldsymbol{\lambda}_{k+1}} \quad & \|\mathbf{x}_{k+1} - \mathbf{x}_k\|\,, \\ \text{s.t} \quad & x_{1,k+1} - x_{1,k} = h(1 + \mathrm{B}_1 z_{k+1}) \\ & x_{2,k+1} - x_{2,k} = h\mathrm{B}_2 z_{k+1} \\ & 0 \in \operatorname{sign}(x_{1,k+1}) + |x_{1,k+1}| - x_{2,k+1}\,, \end{aligned} \tag{4}$$

In this work, we will extend this analysis the more general class of hybrid DAEs that can be formulated as mixed linear complementarity systems. Indeed, in [13], we have seen the constraint (2) can be expressed as mixed linear complementary problems. We show that such representation can be used for the numerical simulation of a wide variety of systems involving hybrid DAEs. We provide both a study of implicit Euler discretization, and numerical simulations of concrete case studies implemented with SICONOS a toolbox for the simulation of non-smooth dynamical systems.

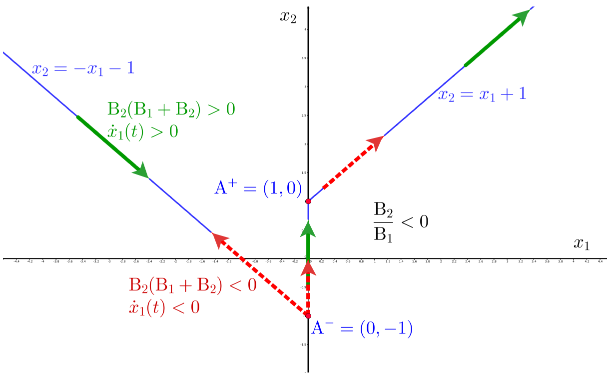

Figure 1: In red and green, we show two possible solutions of the example studied in [13].

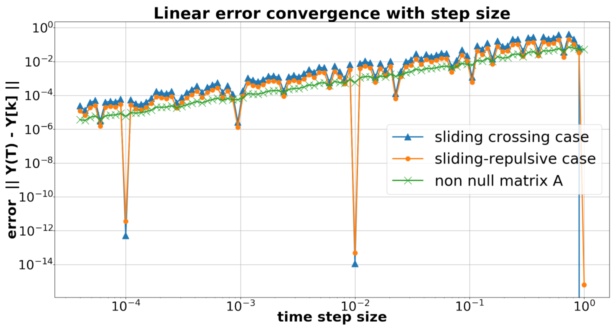

Figure 2: Linear convergence of implicit Euler extension for various cases of the example studied in [13].

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