Multiple impacts in granular chains with Kuwabara-Kono dissipation

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Summary. We study multiple impacts in a chain of beads using the Kuwabara-Kono (KK) contact model [1], a nonsmooth (not Lipschitz continuous) extension of Hertz contact that accounts for viscoelastic damping. For this purpose, we introduce new numerical schemes which approximate dissipative impacts with good accuracy without including the nonsmooth KK viscoelastic component in the contact force. These schemes are derived using the technique of modified equations, which allows us to construct time-discretizations of the nondissipative Hertz law matching numerical dissipation with KK dissipation at different consistency orders. In addition, analytical approximations of wave profiles are derived using asymptotic expansions, in the limit when the exponent of the contact force becomes close to unity. Numerical tests are performed for the simulation of impacts in Newton's cradle and on alignments of free beads.

1 Introduction

In this work, we introduce a new method for the numerical simulation of granular chains which allows to approximate the KK contact model without explicitly including the nonsmooth viscoelastic term in the contact force, *i.e.*, $\gamma = 0$ in (1):

$$f(\delta) = k\left(\delta_+^\alpha + \gamma\,\frac{d}{dt}\delta_+^\alpha\right), \tag{1}$$

where $f(\delta)$ is the spring/dashpot contact force, δ is the indentation, and $\alpha = 3/2$ corresponds to the usual KK model [1]. Our approach is based on the technique of modified (or equivalent) equations, in which effects induced by time-discretization (such as numerical dissipation) are analyzed by considering suitable perturbations of the initial differential equation. In addition, analytical approximations of propagating fronts (generated for example by an impacting piston) are obtained using asymptotic expansions, by considering $\alpha - 1$ as a small parameter.

2 KK numerical dissipation

We consider a Newton's cradle, including local attachments consisting of strings or plates. The dynamical equations including local restoring forces, where the attachment of the nth bead is represented by a spring with linear stiffness K_n, are :

$$m_n\ddot{x}_n = -K_n x_n + k_n\delta_{(n-1)+}^{3/2} - k_{n+1}\delta_{n+}^{3/2}, \quad 1 \le n \le N, \tag{2}$$

with $\delta_n = x_n - x_{n+1}$ and $k_1 = k_{N+1} = 0$. In model (2), collisions are assumed nondissipative. Setting $x = (x_1, \ldots, x_N)$, $v = (\dot{x}_1, \ldots, \dot{x}_N)$ and $y = (x, v)$, system (2) can be reformulated as

$$\dot{y} = f(y) = \begin{pmatrix} v \\ g(x) \end{pmatrix}, \tag{3}$$

where the components of $g(x) \in \mathbb{R}^N$ are given by $g_n(x) = \frac{1}{m_n}\left(-K_n x_n + k_n\delta_{(n-1)+}^{3/2} - k_{n+1}\delta_{n+}^{3/2}\right)$. In order to approximate solutions of (10), we introduce the implicit one-step method

$$\begin{aligned} \frac{X_n^{k+1} - X_n^k}{h} =& \theta V_n^{k+1} + (1-\theta)\, V_n^k, \\ \frac{V_n^{k+1} - V_n^k}{h} =& \frac{1}{m_n}\left[-(1-\mu)\, K_n X_n^{k+1} + \theta\left(k_n\left(\Delta_{n-1}^{k+1}\right)_+^{3/2} - k_{n+1}\left(\Delta_n^{k+1}\right)_+^{3/2}\right)\right] \\ &+ \frac{1}{m_n}\left[-\mu K_n X_n^k + (1-\theta)\left(k_n\left(\Delta_{n-1}^{k}\right)_+^{3/2} - k_{n+1}\left(\Delta_n^{k}\right)_+^{3/2}\right)\right], \end{aligned} \tag{4}$$

where $\Delta_n^k = X_n^k - X_{n+1}^k$, h denotes the time step and $Y_k = \left(X_1^k, \ldots, X_N^k, V_1^k, \ldots, V_N^k\right)$ approximates $y(k\,h)$. We consider a modified (or equivalent) equation (see *e.g.*, [3, 4] and references therein) corresponding to the scheme (4) :

$$\dot{Y} = f(Y) + h\,F_1(Y) + O(h^2), \tag{5}$$

where the coefficient F_1 needs to be determined. After some manipulations the modified equation (5) can be rewritten

$$\dot{X}_n = V_n + h\,(\theta - \frac{1}{2})\,g_n(X) + O(h^2), \tag{6}$$

$$
\begin{aligned}
m_n \ddot{X}_n + K_n \left(h \left(\theta - \mu \right) \dot{X}_n + X_n \right) =& k_n \left(\Delta_{(n-1)+}^{3/2} + 3 \left(\theta - \frac{1}{2} \right) h \Delta_{(n-1)+}^{1/2} \dot{\Delta}_{n-1} \right) \\
& - k_{n+1} \left(\Delta_{n+}^{3/2} + 3 \left(\theta - \frac{1}{2} \right) h \Delta_{n+}^{1/2} \dot{\Delta}_n \right) + O\left(h^2\right),
\end{aligned} \tag{7}
$$

with $\Delta_n = X_n - X_{n+1}$, where a KK dissipation term clearly appears. Setting

$$
\theta = \frac{1}{2} + \frac{\gamma}{2h} = \mu \tag{8}
$$

in (7) and neglecting $O(h^2)$ terms, one recovers the KK model

$$
m_n \ddot{X}_n + K_n X_n = k_n \Delta_{(n-1)+}^{3/2} + \frac{3}{2} \gamma k_n \Delta_{(n-1)+}^{1/2} \dot{\Delta}_{n-1} - k_{n+1} \Delta_{n+}^{3/2} - \frac{3}{2} \gamma k_{n+1} \Delta_{n+}^{1/2} \dot{\Delta}_n. \tag{9}
$$

This observation leads us to the concept of *tailored numerical dissipation*. In this framework, one discretizes the nondissipative model (2) with the dissipative scheme (4)-(8) in order to approximate the dissipative model (9).

3 Analytical approximations of wave profiles

We consider system (9) with a generalized elastic exponent $\alpha > 1$, without local springs (i.e., $K_n = 0$) and for identical beads, so that m_n and k_n are constant and rescaled to unity :

$$
\ddot{X}_n = \Delta_{(n-1)+}^{\alpha} + \alpha \gamma \Delta_{(n-1)+}^{\alpha-1} \dot{\Delta}_{n-1} - \Delta_{n+}^{\alpha} - \alpha \gamma \Delta_{n+}^{\alpha-1} \dot{\Delta}_n. \tag{10}
$$

We consider the limit case of an infinite chain, assume $\Delta_n \geq 0$ (absence of gaps between beads) and set

$$
\Delta_n(t) = \delta \, y(\xi, \tau)^{1/\alpha}, \tag{11}
$$

with $\xi = \varepsilon\,(n - ct)$, $c = \delta^{(\alpha-1)/2}$, $\tau = \frac{\gamma}{2} \varepsilon^2 c^2 t$, and $\varepsilon = \frac{\alpha - 1}{\alpha \gamma c}$. The new variable y can be interpreted as a rescaled force variable. Substituting the Ansatz (11) in the dynamical equations (10), and performing an asymptotic expansion with respect to the small parameter ε, one arrives to a logarithmic Burgers equation

$$
\partial_\tau y + \partial_\xi \left(y \ln y \right) = \partial_\xi^2 y. \tag{12}
$$

The expansion is valid for α close enough to unity, and the logarithmic nonlinearity originates from the approximation $(y - y^{1/\alpha})/(1 - \frac{1}{\alpha}) \approx y \ln y$.
Alternatively, modifying the Ansatz as follows :

$$
\Delta_n(t) = \delta \, y(s, \tau)^{1/\alpha}, \tag{13}
$$

with $s = 2\sqrt{3}\,\varepsilon\,(n - ct)$, $\tau = \sqrt{3}\,\varepsilon^3\,ct$, $\varepsilon = (1 - \frac{1}{\alpha})^{1/2}$ and c defined as above, one obtains the logarithmic KdV-Burgers equation

$$
\partial_\tau y + \partial_s \left(y \ln y \right) + \partial_s^3 y = \mu \, \partial_s^2 y, \tag{14}
$$

where $\mu = c\, 2\sqrt{3}\, \gamma (1 - \frac{1}{\alpha})^{-1/2}$. This expansion is valid when $\alpha \to 1^+$ and assuming $\mu = O(1)$, hence γ has to be small. The amplitude equations (12) and (14) both possess stationary front solutions, which can approximate the asymptotic response of a chain impacted by a piston. They are valid in different regimes regarding the amount of contact damping, and the logarithmic KdV-Burgers equation allows to approximate more general underdamped (oscillatory) fronts.
These approximations are tested by performing direct numerical simulations of the KK model. One observes a convergence of the numerical solutions towards analytical profiles when α is close to unity. In addition, the analytical approximations remain meaningful for $\alpha = 3/2$, i.e. rather far from the asymptotic limit where the amplitude equations have been derived.

4 Conclusions

New approaches for the numerical simulation of granular matter *via* the approximation of the Kuwabara-Kono model by suitable numerical dissipation or the derivation of adapted amplitude equations are presented. More details can be found in [2] and [5].

References

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