Human positioning of a planar pendulum

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Summary. The human positioning of a planar pendulum is investigated. The system is modelled with two particles which are connected with a rope. Collocated proportional-derivative control acts with human reaction delay at the suspension point of the pendulum that can only move horizontally. The Hopf bifurcation analysis of the system is executed which leads to closed form algebraic expressions for the Poincaré-Lyapunov coefficient and for the amplitude of oscillation.

Introduction

The motion of a planar pendulum is examined at its downward position considering human position control. The human intervention is modelled with a proportional-derivative (PD) controller subjected to constant reaction delay, which leads to a system of delay differential equations (DDEs). The inclusion of the time delay implies that the stability region in the PD-plane will be bounded and sub- and supercritical Hopf bifurcations appear along the stability boundary.
The stability and amplitude of the periodic solutions close to the bifurcation point are calculated analytically leading to closed form algebraic equations. We follow the algebraic procedures of Hopf bifurcation calculation as given in [1, 2, 3].

Mechanical model

The coupled hand-held pendulum system is modelled with two point masses connected with a rope as shown in Fig. 1. The human hand is modelled with the mass m_1 on which the control force F acts; this is the pivot point of the planar mathematical pendulum with mass m_2 and length l. The controlled mass can slide in a linear guide so that it can only move along the x axis. The horizontal displacement of the hand and the bottom point of the pendulum are denoted with x_1 and x_2, respectively. The equations of motion of the system are derived with respect to the two generalized coordinates

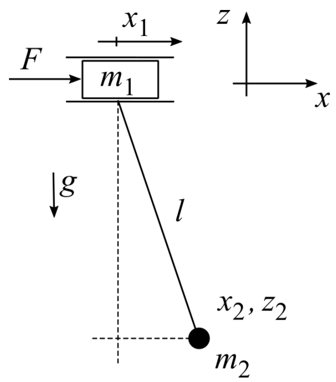

Figure 1: In-plane model of the coupled hand-pendulum system

x_1 and x_2 [4]. These expressions are highly nonlinear, therefore, their Taylor series are calculated up to third order in x_1 and x_2 leading to the governing equations:

$$\ddot{x}_1 = \frac{F}{m_1} + \frac{m_2}{m_1}\frac{g}{l}(x_2 - x_1) - \frac{m_2}{m_1}\left(\left(\frac{m_2}{m_1} + \frac{1}{2}\right)\frac{g}{l^3}(x_2 - x_1)^3 - \frac{1}{l^2}(x_2 - x_1)(\dot{x}_2 - \dot{x}_1)^2 + \frac{F}{m_1 l^2}(x_2 - x_1)^2\right), \quad (1)$$

$$\ddot{x}_2 = -\frac{g}{l}(x_2 - x_1) + \left(\frac{m_2}{m_1} + \frac{1}{2}\right)\frac{g}{l^3}(x_2 - x_1)^3 - \frac{1}{l^2}(x_2 - x_1)(\dot{x}_2 - \dot{x}_1)^2 + \frac{F}{m_1 l^2}(x_2 - x_1)^2, \quad (2)$$

where the time derivative is denoted by dot. The control force can be chosen in different ways. In this study, a collocated PD control is investigated which means that the human operator acts based on the displacement and velocity of his/her hand:

$$F(t) = -Px_1(t - \tau) - D\dot{x}_1(t - \tau). \quad (3)$$

Here, τ stands for the delay caused by the reaction time, while P and D are the proportional and derivative gain parameters, respectively.

Stability analysis

Introduce the dimensionless distances $\tilde{x}_i = x_i/l$ (for $i = 1, 2$) and the dimensionless time $\tilde{t} = t/\tau$, furthermore, the transformed characteristic exponents and angular frequencies $\tilde{\lambda} = \tau\lambda$ and $\tilde{\omega} = \tau\omega$, respectively. By abuse of notation we drop the tildes and introduce the dimensionless parameters: $\mu = m_2/m_1$, $\alpha = \tau\sqrt{g/l}$, $p = P\tau^2/m_1$, and $d = D\tau/m_1$. Then the characteristic equation assumes the form:

$$D(\lambda) = \lambda^4 + d\lambda^3 e^{-\lambda} + \alpha^2(1+\mu)\lambda^2 + p\lambda^2 e^{-\lambda} + \alpha^2 d\lambda e^{-\lambda} + \alpha^2 p e^{-\lambda} = 0. \tag{4}$$

When the real characteristic exponent crosses the origin at $\lambda = 0$, saddle-node bifurcation appears at $p = 0$. Hopf bifurcation occours when a pair of complex conjugate roots lies on the imaginary axis at $\lambda = \mathrm{i}\omega$, which yields the stability boundary curve:

$$p_{cr} = \left(1 - \frac{\mu\alpha^2}{\omega^2 - \alpha^2}\right)\omega^2\cos\omega, \quad d_{cr} = \left(1 - \frac{\mu\alpha^2}{\omega^2 - \alpha^2}\right)\omega\sin\omega. \tag{5}$$

Let the proportional gain p be the bifurcation parameter. Then the real part of the root tendency $\lambda' = \mathrm{d}\lambda(p_{cr})/\mathrm{d}p$ assumes the form:

$$\Re\left(\lambda'\right) = \frac{\omega}{4\left(a^2+b^2\right)}\left(\sin\omega + \omega\cos\omega + \mu\alpha^2\frac{\omega^2+\alpha^2}{\left(\omega^2-\alpha^2\right)^2}\sin\omega - \frac{\omega\mu\alpha^2}{\omega^2-\alpha^2}\cos\omega\right), \tag{6}$$

where a and b can be expressed as

$$a = \frac{1}{2}\omega\left(1 - \frac{\mu\alpha^2}{\omega^2-\alpha^2}\right)\left(\frac{1}{2}\sin(2\omega) - \omega\right), \quad b = \omega + \frac{\omega\mu\alpha^4}{\left(\omega^2-\alpha^2\right)^2} - \frac{1}{2}\left(1 - \frac{\mu\alpha^2}{\omega^2-\alpha^2}\right)\sin^2\omega^2. \tag{7}$$

The Poincaré-Lyapunov coefficient Δ and the amplitude of oscillation A are obtained in the form:

$$\Delta = \frac{\mu}{8\left(a^2+b^2\right)}\left(\frac{\omega^2}{\omega^2-\alpha^2}\right)^4\left(1 - \frac{\mu\alpha^2}{\omega^2-\alpha^2}\right)\left(\frac{1}{4}\omega\sin(2\omega) - \frac{1}{2}\omega^2\right)\left(2\omega^2 - \frac{3}{2}\alpha^2\right), \tag{8}$$

$$A = \sqrt{-\frac{\Re\left(\lambda'\right)}{\Delta}(p - p_{cr})} = \left(\frac{\omega^2-\alpha^2}{\omega^2}\right)^2\sqrt{-\frac{2}{\mu}\frac{\sin\omega\left(1+\mu\frac{\omega^2\alpha^2+\alpha^4}{(\omega^2-\alpha^2)^2}\right)+\omega\cos\omega\left(1-\mu\frac{\omega\alpha^2}{\omega^2-\alpha^2}\right)}{\left(1-\mu\frac{\alpha^2}{\omega^2-\alpha^2}\right)\left(\frac{1}{4}\omega\sin(2\omega)-\frac{1}{2}\omega^2\right)\left(2\omega^2-\frac{3}{2}\alpha^2\right)}(p-p_{cr})}. \tag{9}$$

The first two terms of Eq. 8 are nonnegative, therefore the last three terms determine the sign of Δ and so the sense of the bifurcation. Fig. 2 shows the saddle node and Hopf bifurcation curves of the system indicating the sub- or subcritical nature as well. The boundary of the stable region is always supercritical.

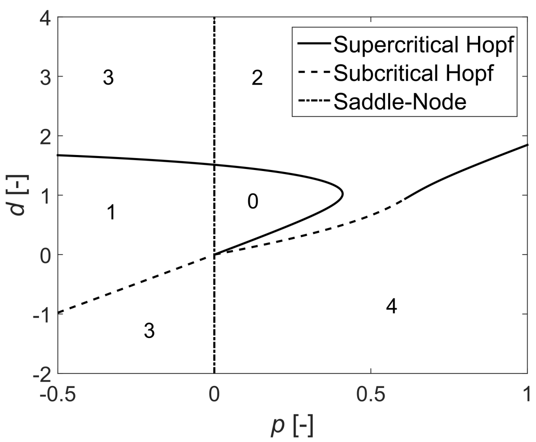

Figure 2: Stability chart in the PD plane. The numbers indicate the number of unstable characteristic exponents. ($\mu = 1/30$, $\alpha = 1.1437$)

Conclusions

The model can explain why hand-held pendulums often oscillate in spite of the intention of the human operator.

References

[1] Hassard B. D., Kazarinoff N. D., Wan Y.-H. (1981) Theory and Applications of Hopf Bifurcation, *London Mathematical Society Lecture Note Series*, vol. 41. Cambridge University Press, Cambridge

[2] Stepan G. (1989) Retarded Dynamical Systems: Stability and Characteristic Functions, *Pitman Research Notes in Mathematics*, vol. 210. Longman, Essex

[3] Orosz G. (2004) Hopf bifurcation calculations in delayed systems, *Periodica Polytechnica Ser. Mech. Eng.* **48**:189-200

[4] Chin C., Nayfeh A. H., Abdel-Rahman E. (2001) Nonlinear dynamics of a boom crane, *Journal of Vibration and Control* **7**:199-220