Model order reduction approach for problems with moving discontinuities

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Summary. We propose a new model order reduction (MOR) approach to obtain effective reduction in the context of transport-dominated problems or hyperbolic partial differential equations. The main ingredient is a novel decomposition of the solution into *(i)* a function that tracks the evolving discontinuity and *(ii)* the residual part that is devoid of shock features. This decomposition strategy is then combined with Proper Orthogonal Decomposition which is applied to the residual part only to develop an efficient reduced-order model representation for problems with multiple moving and possibly merging discontinuous features. Numerical case-studies show the potential of the approach in terms of computational accuracy compared with standard MOR techniques.

Problem Description

Hyperbolic partial differential equations (PDEs) are ubiquitous in science and engineering. Applications encompassing the fields of chemical industry, nuclear industry, drilling industry, etc., fall within this class. Moving discontinuities (such as shock-fronts) are representative features of this class of problems and pose a major hindrance to obtain effective reduced-order model representations [1]. As a result, standard Model Order Reduction techniques [2] do not fit the requirements for real-time estimation and control or multi-query simulations for such problems. This motivates us to investigate and propose efficient, advanced and automated approaches to obtain reduced models, while still guaranteeing the accurate approximation of wave propagation (and wave interaction) phenomena.
The main contribution of the work is to propose a new decomposition ansatz that decomposes the solution into *(i)* a basis function that tracks the evolving discontinuity (wave front) and *(ii)* the residual part that is expected to be devoid of shock features. This decomposition renders the residual part to be amenable for efficient basis generation. We, then, use these generated bases to apply Proper Orthogonal Decomposition (POD) on the residual part and later reconstruct the solution by lifting it to the high-dimensional problem space. We finally assess the combined performance of decomposition, reduction and reconstruction approach (as opposed to conventional reduction and reconstruction approach) in the scope of transport-dominated problems with moving and interacting discontinuities.

Mathematical Formulation

We discuss the proposed decomposition ansatz and outline the whole procedure in order to obtain a reduced-order model.

Decomposition step

We consider a scalar 1D conservation equation of the form:

$$\partial_t u(x,t) + \partial_x f(u(x,t)) = 0, \;\; u(x,0) = u_0(x). \tag{1}$$

We assume that $u(x,0) = u_0(x)$ already has S number of shocks at locations $x_1(0), ..., x_S(0)$ with values $u^-(x_s(0),0)$, $s = 1, ..., S$ from the left and values $u^+(x_s(0),0), \quad s = 1, ..., S$ from the right. We associate a single basis function $\sigma_s(x - x_s(t))$ to each discontinuity at their respective locations. This basis function has a jump of height 1, i.e., $\sigma_s^+(0) - \sigma_s^-(0) = 1$, at the location of the discontinuity. We now decompose the solution, u, in the following way:

$$u(x,t) = \sum_{s=1}^{S} j_s(t)\sigma_s(x - x_s(t)) + u_r(x,t),$$

$$j_s(t) = u^-(x_s(t),t) - u^+(x_s(t),t). \tag{2}$$

When $x_s(t)$ exactly matches the location of the shocks and (2) is exactly fulfilled, then $u_r(x,t)$ represents a function without any shocks (discontinuities) and hence is amenable to a low-rank approximation.
The time-stepping scheme can be defined in the following manner. In each time step, we first compute updated shock locations $x_s(t^{n+1})$, jumps $j_s(t^{n+1})$ and then compute the residual part $u_r(x,t^{n+1})$ from

$$u_r(x,t^{n+1}) - u_r(x,t^n) = \sum_{s=1}^{S} j_s(t^n)\sigma_s(x - x_s(t^n)) - \Delta t \partial_x f(u(x,t^n)) - \sum_{s=1}^{S} j_s(t^{n+1})\sigma_s(x - x_s(t^{n+1})). \tag{3}$$

Reduced Order Model

The standard way to construct a reduced-order model (ROM) is to reduce (1) by applying Galerkin projection on u. Instead, we apply POD on the residual part, i.e., we reduce (3) via Galerkin projection onto $V_N \subseteq V_h$ (where V_N is a reduced space and V_h is a high-fidelity space). Upon considering the projection operator $P_N : V_h \to V_N$, the reduced scheme on the residual part is:

$$u_{r,N}^{k+1} = u_{r,N}^{k} + P_N\Big(\sum_{s=1}^{S} j_{s,N}(t^k)\sigma_s(x - x_{s,N}(t^k)) - \Delta t \partial_x f(P_N^{'} u_N^k) - \sum_{s=1}^{S} j_{s,N}(t^{k+1})\sigma_s(x - x_{s,N}(t^{k+1}))\Big), \quad (4)$$

where k indicates the time-instant, $u_{r,N}^k \in V_N$, $u_{r,N}^0 = P_N(u_r^0)$ with u_N^k defined as:

$$P_N^{'} u_N^k = \sum_{s=1}^{S} j_{s,N}(t^k)\sigma_s(x - x_{s,N}(t^k)) + P_N^{\prime} u_{r,N}^k. \quad (5)$$

Numerical Experiments

We reduce the Burgers equation given by: $\partial_t u + \partial_x(\frac{u^2}{2}) = 0, x \in [0, L]$, for illustrating the potential of the proposed approach. The case studies under consideration assume that the shock is already present in the initial data, which for single and multiple wavefront scenarios, is respectively given by:

$$u(x,0) = u_0(x) = \begin{cases} x, & 0 \le x \le 1, \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad u(x,0) = u_0(x) = \begin{cases} x-2, & 2 \le x \le 4, \\ (x-5)/2, & 5 \le x \le 7, \\ 0, & \text{otherwise.} \end{cases}$$

We consider the spatial domain to be of length $L = 10$ and use an upwind finite volume scheme for the spatial discretization and a first-order Forward Euler for the time-stepping. We take 8000 steps in time for the scenarios under consideration i.e., $t \in [0, 4]$ with a timestep of 0.0005. We consider three different spatial mesh resolutions (spatial step size of 0.005, 0.002 and 0.001) to assess the performance of the standard (POD without decomposition) and the proposed approach.

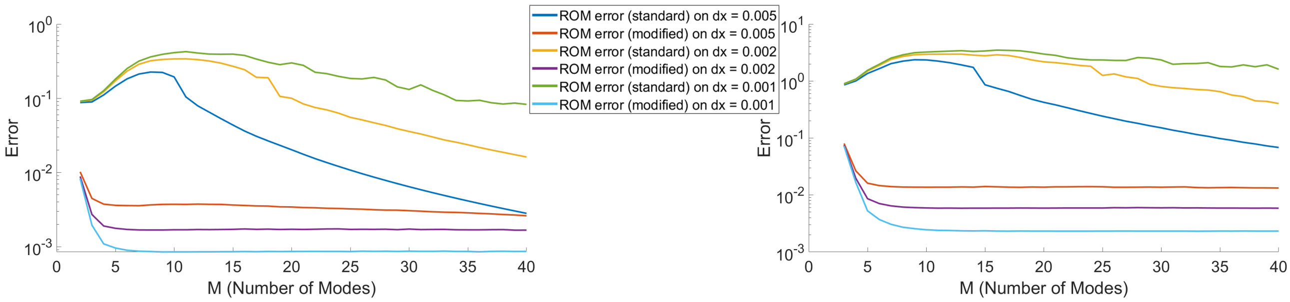

Figure 1: ROM error for the single wavefront scenario (left) and multiple wavefront scenario (right) under different mesh sizes.

We consider that $j_{s,N} = j_s$ and $x_{s,N} = x_s$ and use these jumps and shock locations during the ROM time-stepping. We, further, use the computed residual part to generate the bases and build a ROM. Figure 1 demonstrates the decay of the 'L^2 in space and L^2 in time' (absolute) error (between the full-order model governed by (1) and the reconstruction given by (5)) for the scenarios of interest. Firstly, the initial error incurred via the proposed approach is lower than that of the standard approach. This is attributed to the fact that our decomposition approach associates a basis function corresponding to the travelling discontinuity. Secondly, the rate of decay of the error is better for the proposed approach compared to the standard approach. Moreover, the ROM error for the proposed approach stagnates later for finer mesh-sizes. It is also observable that the mesh refinement reduces the ROM error obtained via proposed approach in contrast to the ROM error obtained via standard approach. The difference in the order of magnitude of the ROM error (at a certain number of basis function) computed via standard and proposed approach is even more pronounced for fine mesh-sizes.

Conclusions

We have proposed a decomposition ansatz for problems with moving discontinuities and combined it with POD applied to the residual part only. We have show-cased the performance of the proposed approach on the Burgers equation. The proposed approach is able to resolve the discontinuities and also offers reduction in ROM error by using much less number of basis functions. We will perform numerical experiments on many other mathematical models (falling within the class of transport-dominated problems) in the near future.

References

[1] M. Ohlberger and S. Rave, Reduced basis methods: Success, limitations and future challenges," Proceedings of the Conference Algoritmy, 2016.

[2] A. Quarteroni and G. Rozza, edts. Reduced Order Methods for Modeling and Computational Reduction. Springer International Publishing, Cham, 2014.