On periodic solutions and modal energy transfer of mechanical systems with state-dependent impulsive stiffness excitation

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Summary. In the present contribution, mechanical systems with impulsive stiffness excitation are investigated. It is shown that periodic solutions exist, which result in a repeated transfer of vibration energy from lower to higher modes and vice-versa. This allows that the structural damping of the mechanical system can be utilized more effectively, resulting in a faster decay of transient vibrations compared to the case where no impulsive excitation is present.

Introduction

The transfer of vibration energy, either in the modal space, or spatially to an attached system, allows to reduce transient vibrations after some initial disturbance. In the first case the enhanced damping properties of higher modes can be utilized more effectively, see. e.g. [1], whereas in the second case energy is transferred in a one-way, irreversible manner to a nonlinear coupled additional system, denoted as nonlinear energy sink (NES), see [2], for example.
In the present contribution, mechanical systems with state-dependent impulsive stiffness excitation are investigated. It is shown that, in the conservative case, periodic solutions exist which result in a periodic exchange of vibration energy across modes. By taking structural damping into account, the effect on the total energy content of mechanical systems is demonstrated.

Periodic solutions and energy transfer

In the following, mechanical systems with impulsive stiffness excitation described by equations of motion of the form

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + (\mathbf{K} + \sum_{k=1}^{K} \varepsilon_k(\mathbf{x}_k, \dot{\mathbf{x}}_k)\mathbf{G}\delta(t-t_k))\mathbf{x}(t) = \mathbf{0}, \tag{1}$$

are investigated. Therein, $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ represent the constant $(n \times n)$-dimensional mass-, damping- and stiffness-matrix. Impulsive parametric excitation is introduced at equidistant instants of time t_k by using Dirac-delta functions $\delta(t-t_k)$, where the state-dependent strength of the impulses is denoted as $\varepsilon_k(\mathbf{x}_k, \dot{\mathbf{x}}_k)$. It was shown in [3], that the state of the system $\mathbf{r}(t) = [\mathbf{x}(t)\ \dot{\mathbf{x}}(t)]^T$ just after an impulse at t_k, i.e. at $t_{k,+}$, can be related to the state after the preceding impulse at $t_{k-1,+}$ by

$$\mathbf{r}(t_{k,+}) = \mathbf{J}_k(\varepsilon_k)\boldsymbol{e}^{\mathbf{A}\Delta T}\mathbf{r}(t_{k-1,+}), \tag{2}$$

where

$$\mathbf{J}_k = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\varepsilon_k\mathbf{M}^{-1}\mathbf{G} & \mathbf{I} \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix}, \tag{3}$$

and $\Delta T = t_k - t_{k-1}$ holds. The matrix $\mathbf{J}_k$ was denoted as jump-transfer matrix by *Hsu*, see [3]. If the impulsive strength ε_k is selected to be state-dependent according to

$$\varepsilon_k = (\sum_{i=1}^{n} \mathbf{g}_i^T\mathbf{x}_{k-}\dot{x}_{i,k-})/(\frac{1}{2}\sum_{i=1}^{n}(\mathbf{g}_i^T\mathbf{x}_{k-})^2/m_i), \tag{4}$$

see [4], neither energy is extracted from, nor fed to the mechanical system by an impulse, i.e. the impulse is energy-neutral. In this case, $\mathbf{J}_k$ becomes a constant matrix and Eqn. (2) can be written in the form

$$\mathbf{r}(t_{k,+}) = \underbrace{\mathbf{J}\boldsymbol{e}^{\mathbf{A}\Delta T}}_{\mathbf{Q}(\Delta T)}\mathbf{r}(t_{k-1,+}) = \mathbf{Q}^k(\Delta T)\mathbf{r}_0 = \boldsymbol{\Psi}\boldsymbol{\Lambda}^k\boldsymbol{\Psi}^{-1}\mathbf{r}_0, \tag{5}$$

where $\mathbf{r}_0 = \mathbf{r}(t_0 = 0)$. The matrix $\boldsymbol{\Psi}$ is comprised of the eigenvectors of $\mathbf{Q}$, and $\boldsymbol{\Lambda} = \mathrm{diag}(\lambda_i),\ i = 1, 2, \dots 2n$, of the corresponding eigenvalues. Following the notation in [5], a periodic solution with period R is given by a sequence of R distinct points $\mathbf{r}^*$ in the state-space according to

$$\mathbf{r}^*(t_{m+r,+}) = \mathbf{Q}^r\mathbf{r}^*(t_{m,+}), \quad r = 1, 2, \dots R-1, \tag{6}$$

$$\mathbf{r}^*(t_{m+R,+}) = \mathbf{Q}^R\mathbf{r}^*(t_{m,+}) = \mathbf{r}^*(t_{m,+}), \tag{7}$$

denoted as P-R solution. It can be seen from Eqn. (5), that the last condition (Eqn. (7)) is fulfilled if there exists a timespan ΔT between adjacent impulses, for which $\boldsymbol{\Lambda}^R = \mathbf{I}$ holds, where $\mathbf{I}$ represents the identity matrix. With a simple example, the existence of such cases is demonstrated in the following.
The investigated mechanical system comprises two masses connected by stiffness and damping elements and is pinned on one end, see Fig. (1), where the stiffness $k_{01} = \bar{k}_{01} + \varepsilon_k g_{01}\delta(t-t_k)$, i.e. consists of a constant and an impulsive part. At equidistant instants of time t_k, stiffness impulses with a strength according to Eqn. (4) are applied. Hence, the equations of motion are of the form of Eqn. (1). As system parameters $m_1 = 1$, $m_2 = 0.5$, $\bar{k}_{01} = 1$, $k_{12} = 2$, and a stiffness-proportional damping $\mathbf{C} = \alpha\mathbf{K}$, $\alpha = 0.01$, were used for the numerical calculations.

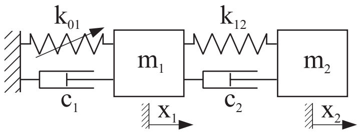

Figure 1: Sketch of investigated mechanical system.

Figure (2a) shows the eigenvalues λ_i, $i = 1, \dots 4$, of the mapping matrix $\mathbf{Q}$ in the undamped case ($\alpha = 0$), for different values of the timespan ΔT between adjacent impulses. One notes that within the investigated interval of ΔT, two real and two complex conjugate eigenvalues exist. Exemplarily, the case $\Delta T = 6.68672$ where $\lambda_{12} = \pm 1$ and $\lambda_{34} = \pm i$, see the dotted unit-circle in Fig. (2a), is investigated in more detail. Applying the mapping matrix $\mathbf{Q}$ four times gives $\lambda^R_{1,2} = 1$ and $\lambda^R_{3,4} = 1$, i.e. the mechanical system attains the initial state again after applying four impulses. Hence, we have a P-4 solution with a period of $4\Delta T_{P-4}$. Figure (2b) depicts the corresponding 4-periodic timeseries of the impulsive strength ε_k. The effect of the impulsive excitation on the modal coordinates y_1 and y_2 is shown in Fig. (2c). As a first mode deflection according to $\mathbf{y}(t = 0) = [1\ 0]^T$ and $\dot{\mathbf{y}}(t = 0) = [0\ 0]^T$ was used as initial condition, the mechanical system does not show any second mode vibrations initially. This changes with the application of the first impulse, as energy is transferred from the first to the second mode. In the following, the system exhibits first and second mode vibrations simultaneously, until after one period the energy content of the second mode vanishes again. Thereafter, one observes a periodic exchange of energy from the first to the second mode and vice-versa. Including structural damping ($\alpha = 0.01$) allows to exploit the enhanced damping properties of the second mode, see Fig. (2d). During the phases where the second mode contains vibration energy, the total energy content E of the mechanical system decreases faster compared to the phases where only first mode vibrations occur. This results in a globally faster decay of the vibration energy E compared to E_0 (energy content where no impulsive excitation is present).

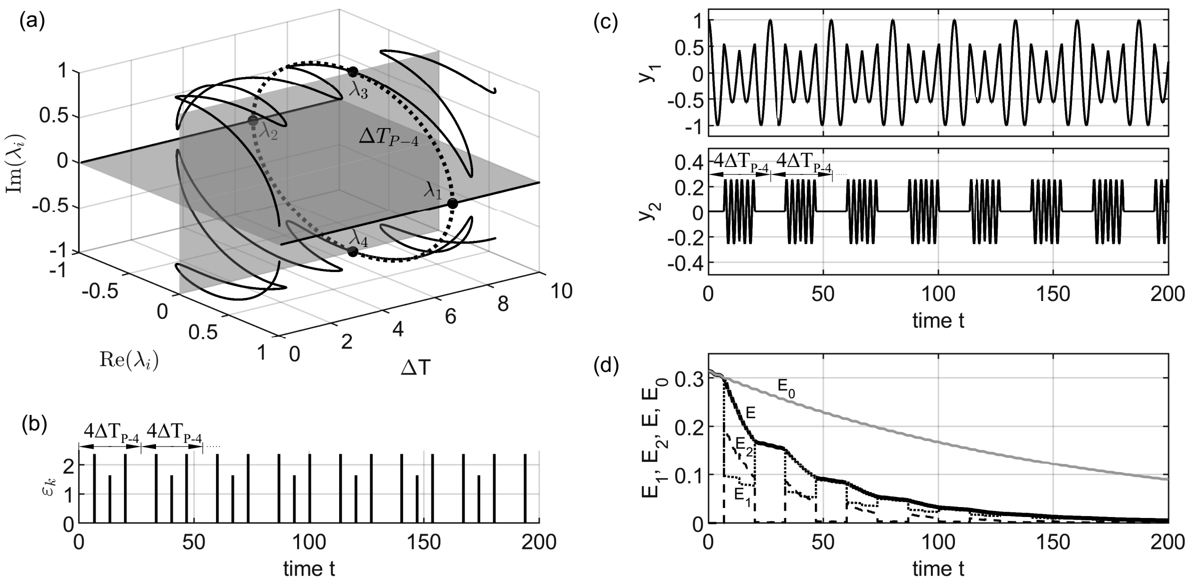

Figure 2: Eigenvalues λ_i, $i = 1, \dots 4$, of mapping matrix $\mathbf{Q}$ for different values of the timespan ΔT between adjacent impulses - undamped case $\alpha = 0$, (a). Impulsive strength (b), and modal displacements y_1 and y_2 (c) for the case $\Delta T = \Delta T_{P-4} = 6.68672$ and $\alpha = 0$. Modal energy contents E_1 (dotted) and E_2 (dashed), total energy content of the mechanical system E, and total energy content E_0 of system without impulsive excitation for the damped case (d).

Conclusions

It was demonstrated that conservative mechanical systems exhibited to impulsive stiffness excitation of energy-neutral kind can show a periodic behaviour, which can be utilized effectively to enhance the damping of transient vibrations.

References

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