Dynamics of Interacting Populations in a Bounded Domain: Control and Estimation under Nonlinearity and Uncertainty

Tatiana F. Filippova
Department of Optimal Control, Krasovskii Institute of Mathematics and Mechanics, Ural Branch of Russian Academy of Sciences, Ural Federal University, Ekaterinburg, Russian Federation

Summary. The nonlinear control systems of Lotka - Volterra type which describe the dynamics of the interaction of predators and their preys are studied. It is assumed that the initial states of the system are not known precisely but belong to a given ellipsoid in the state space and there is no additional information (for example, probabilistic, statistical, etc.) on unknown values. We find the external ellipsoidal estimates of corresponding reachable sets for control systems under study. The considered models may describe the behavior of competing firms, population growth, environmental change, development of individual industries, etc. The results of modeling based on proposed methods are included to illustrate the main proposed ideas and presented estimation algorithms.

Introduction

The problem of state estimation for control systems under conditions of nonlinearity and uncertainty is studied. The case is investigated here when the probabilistic data of noise and possible errors is not available, and only some restriction on unknown parameters and functions are given. Models of this kind may be found in many applied problems including physics, economics, biology, ecology, etc. As one of the important and key idea to study such problems we use the so-called set-membership approach which was developed earlier [1, 2, 3, 4] to deal with a whole set of feasible parameters, consistent with the model structure, for systems with bounded uncertainty, with special types of measurements characterization etc.

The solutions of such classes of control and estimation problems with set-membership uncertainty are based on the construction and on the analysis of the corresponding reachable sets or their analogs. In this paper we study the procedures of upper estimating reachable sets for nonlinear control systems of Lotka-Volterra type. We use here the ideas and results of state estimation theory developed for nonlinear control systems which have a special quadratic dynamical structure [5, 6, 7, 8] and several new schemes of the problem solution are developed here. We prove here theoretical results and formulate related numerical algorithms for constructing external ellipsoidal estimates of reachable sets for nonlinear uncertain control systems of the studied type. Numerical examples and results of related simulations are included to illustrate the basic ideas and results.

Problem statement

The paper deals with the problems of control and state estimation for a dynamical control system

$$\dot{x}(t) = A(t)x(t) + f(x(t)) + G(t)u(t), \;\; x \in R^n, \;\; t_0 \le t \le T, \tag{1}$$

with unknown but bounded initial state

$$x(t_0) = x_0, \;\; x_0 \in X_0, \;\; X_0 \subset R^n, \tag{2}$$

and with control constraint

$$u(t) \in U, \;\; U \subset R^m, \;\; \text{for a.e. } t \in [t_0, T]. \tag{3}$$

Here matrices $A(t)$ and $G(t)$ (of dimensions $n \times n$ and $n \times m$, respectively) are assumed to be continuous on $t \in [t_0, T]$, X_0 and U are compact and convex. The nonlinear n-vector function $f(x)$ in (1) is assumed to be of quadratic type

$$f(x) = (f_1(x), \dots, f_n(x)), \;\; f_i(x) = x' B_i x, \; i = 1, \dots, n, \tag{4}$$

where B_i $(i = 1, \dots, n)$ are constant $n \times n$ - matrices.

In previous studies [5, 6, 7, 8], it was taken as the main assumption that matrices B_i in (1)–(4) are positive definite; this additional condition simplified the analysis of nonlinearity in studying the structure and the properties of reachable sets of the control system (1)–(4).

As a particular kind of the above control problem we consider here the following Lotka-Volterra system which describes the classical ecological predator-prey (or parasite-host) model with additional control functions:

$$\begin{cases} \dot{x_1}(t) &= \;\; ax_1 - bx_1x_2 + u_1, \\ \dot{x_2}(t) &= \;\; -cx_2 + dx_1x_2 + u_2, \end{cases} \qquad x(t_0) = x_0, \;\; t_0 \le t \le T. \tag{5}$$

Here we assume that numbers $a, b, c, d > 0$ are given and initial vectors x_0 are unknown but bounded, that is we have the inclusion $x_0 \in X_0$, where X_0 is a given compact subset of R^2. This assumption may be interpreted for example in such a way that we do not know exactly the initial states (or amounts) of predators and prey. We assume also that controls $u(t)$ in (3)-(5) are taken measurable in Lebesgue on $[t_0, T]$, also the inclusion is true

$$u(t) \in U, \quad \text{a.e. } t \in [t_0, T], \tag{6}$$

where $U \in compR^2$. The choice of a control can influence, in particular, the rate of change in amounts of predators and prey.

Basing on results of ellipsoidal calculus [1, 4] and estimation schemes described in [5, 6, 8] we present here the modified state estimation approaches which use the special structure of nonlinearity of studied control system (1)–(6) and combine advantages of estimating tools mentioned above. Numerical simulation schemes together with modeling examples are also included.

Main results

Note first that we have in the system (5)-(6) a bit more simple situation than in general case of the system (1)-(4), namely we have the equality $f(x)' = x_1 x_2 \cdot (-b, d)$. The idea used here in the analysis of reachable sets of the system (5)–(6) is based on the following transformion of the system (1)–(4) to the new one which will include only positive definite quadratic forms, this case is more convinient for the analysis and for further numerical modeling. So we consider the following modified control system

$$\begin{gathered} \dot{z} = A^* z + f_\varepsilon^{(1)}(z) \cdot d^{(1)} + f_\varepsilon^{(2)}(z) \cdot d^{(2)} + w(t), \\ z_0 \in \mathcal{Z}_0, \quad w \in \mathcal{W}, \quad t_0 \le t \le T, \end{gathered} \tag{7}$$

with

$$A^* = \begin{pmatrix} A & -C \\ -C & A \end{pmatrix}$$

and with functions $f_\varepsilon^{(1)}(z)$ and $f_\varepsilon^{(2)}(z)$ being the positive definite quadratic forms with matrices $B_\varepsilon^{(1)} = diag\{1, \varepsilon^2\}$ and $B_\varepsilon^{(2)} = diag\{\varepsilon^2, 1\}$, respectively.

We can find now the external ellipsoidal estimates of reachable set $Z(t)$ of the system (7) applying for this purpose results of [7]. The case of the presence of a state constraint on the dynamics of the dynamical system is considered separately; here, the possibility of applying the procedure for removing restrictions using the results of [2] is discussed.

We formulate also a numerical discrete-time algorithms of ellipsoidal estimating the reachable sets of the studied uncertain system. Examples and numerical simulation results related to procedures of set-valued approximations of trajectory tubes and reachable sets are also presented.

Conclusions

The paper deals with the problems of control and state estimation for a dynamical control system with unknown but bounded initial state. The solution to the differential control system is studied through the techniques of trajectory tubes of the theory of differential inclusions. The estimation approach uses the special nonlinear structure of the Lotka-Volterra system. Examples and numerical results related to state estimation procedures of reachable sets are presented.

The research was supported by the Russian Foundation for Basic Researches (RFBR) under Project 18-01-00544a.

References

[1] Kurzhanski A. B., Valyi I. (1997) Ellipsoidal Calculus for Estimation and Control. Birkhauser, Boston.

[2] Kurzhanski A. B., Filippova T. F. (1993) On the Theory of Trajectory Tubes – a Mathematical Formalism for Uncertain Dynamics, Viability and Control. *Advances in Nonlinear Dynamics and Control: a Report from Russia, Progress in Systems and Control Theory, (ed. A.B. Kurzhanski)*, Birkhauser, Boston. V. 17:122-188.

[3] Schweppe F. (1973) Uncertain Dynamic Systems. Prentice-Hall, Englewood Cliffs, New Jersey.

[4] Chernousko F. L. (1994) State Estimation for Dynamic Systems. CRC Press, Boca Raton.

[5] Filippova T. F. (2014) Asymptotic Behavior of the Ellipsoidal Estimates of Reachable Sets of Nonlinear Control Systems with Uncertainty. *Proc. of the 8th European Nonlinear Dynamics Conference (ENOC 2014)*. Vienna, Austria, July 6-11, 2014, H. Ecker, A. Steindl and S. Jakubek (Eds.), Institute of Mechanics and Mechatronics, TU-Vienna, Austria. CD-ROM volume (ISBN: 978-3-200-03433-4). Paper-ID 149. Vienna, 2014.

[6] Filippova T. F. (2017) Differential Equations of Ellipsoidal State Estimates for Bilinear-Quadratic Control Systems under Uncertainty. *J. Chaotic Modeling and Simulation (CMSIM)*, **1**:85–93, 2017.

[7] Filippova, T. F. (2017). Ellipsoidal estimates of reachable sets for control systems with nonlinear terms. *IFAC-PapersOnLine: 20th World Congress of the International Federation of Automatic Control (IFAC-2017), Toulouse, France, July 9-14, 2017: proceedings*, **50(1)**:15925–15930.

[8] Filippova T.F. (2019) Description of Dynamics of Ellipsoidal Estimates of Reachable Sets of Nonlinear Control Systems with Bilinear Uncertainty. *Lecture Notes in Computer Science. 2019. Vol. 11189 (Nikolov G., Kolkovska N., Georgiev K. (eds) Numerical Methods and Applications, NMA 2018)*. Springer, Cham, 97–105.