Error estimates for model order reduction of Burgers' equation

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Summary. Burgers' equation is a nonlinear scalar partial differential equation, commonly used as a testbed for many newly developed model order reduction techniques and error estimates. Model order reduction of the parameterized Burgers' equation is commonly done by the Reduced Basis (RB) method. In this method, an error estimate plays a crucial rule in accelerating the offline phase (when the reduced model is built) and also in quantifying the error induced after reduction in the online phase (when the reduced model is used to find fast solutions). In this study, we introduce two new estimates for this reduction error. The first error estimate is based on the Lur'e-type model formulation of the system obtained after the full-discretization of Burgers' equation. The second error estimate is built upon snapshots generated in the offline phase of the RB method. The second error estimate is applicable to a wider range of systems compared to the first error estimate. Results reveal that when conditions for the error estimates are satisfied, the error estimates are accurate and work efficiently in terms of computational effort.

Problem description

One of the simplest and yet fundamental nonlinear equations describing a conservative system is Burgers' equation, which is sometimes referred to as the scalar version of the Navier-Stokes equations [1]. This equation is defined in the infinite- and finite- dimensional setting, obtained after Finite Volume (FV) discretization, as follows:

$$\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}\left(f(u)\right)=0,\ t\in[0,T],\ x\in[0,L] \quad \xrightarrow{\text{FV}} \quad \begin{aligned}&\Sigma_{lin}:\begin{cases}\boldsymbol{U}^{n+1}=L_{lin}\boldsymbol{U}^n+BU_0^n-\dfrac{\Delta t}{4\Delta x}L_{nl}\boldsymbol{U}_{nl}^n+\dfrac{\Delta t}{2\Delta x}B(U_0^n)^2,\\ \quad y^n=C_y\boldsymbol{U}^n,\\ \quad z^n=\boldsymbol{U}^n,\end{cases}\\ &\Sigma_{nl}:\boldsymbol{U}_{nl}^n=g(z^n):=(z^n)^2.\end{aligned} \tag{1}$$

where $u:=u(t,x;\boldsymbol{\mu})$ is the conservative variable of the system, $\boldsymbol{U}^n$ is the vector of the conservative variables computed at the grid cells at time index n, U_0^n is the boundary input at time index n, y^n is the output of interest specified by matrix C_y, and finally $f(u):=u^2/2$ is the flux function associated with Burgers' equation. Here, t represents time and T is the time horizon of the simulation. In addition, x denotes the spatial coordinate and L is the length of the spatial domain. Finally, $\boldsymbol{\mu}\in\mathcal{D}$ is a vector of parameters used in (1) that varies in the multi-query analysis within the parameter domain $\mathcal{D}\in\mathbb{R}^R$, with $R>0$ the number of varying parameters. We assume that the initial condition and the boundary condition are represented by the varying parameters. For the initial condition, we assume $u(0,x;\boldsymbol{\mu})=\mu_1$, which is constant over the spatial domain. For the boundary condition at $x=0$, we assume

$$u(t,0;\boldsymbol{\mu})=\begin{cases}\mu_1, & t=0,\\ \mu_2, & t>0.\end{cases} \tag{2}$$

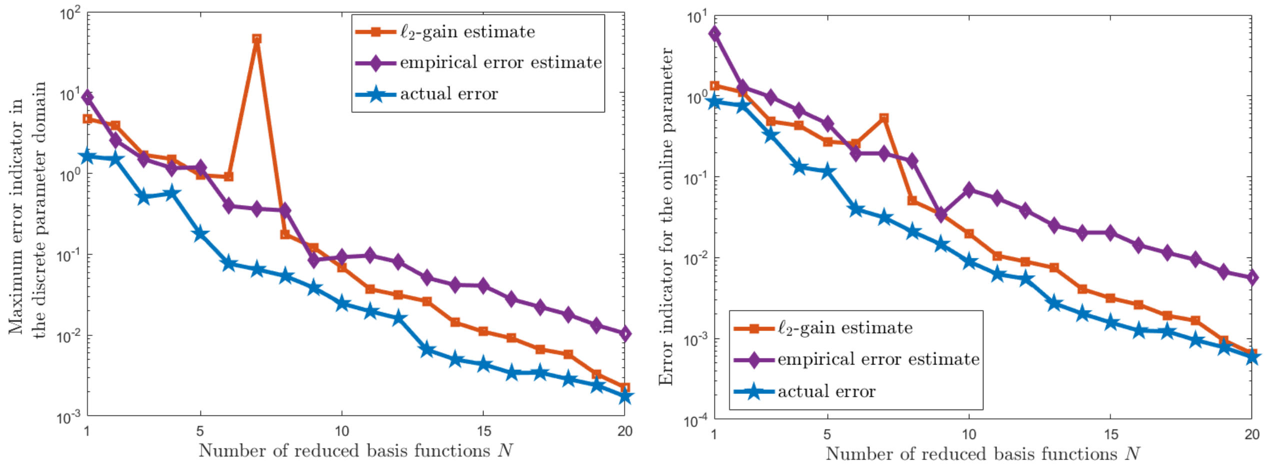
Therefore, in this study, we have $\boldsymbol{\mu}=[\mu_1,\mu_2]$.
The fully discretized system $(\Sigma_{lin},\Sigma_{nl})$ as in (1) usually has a large dimension. Therefore, real-time simulations cannot be achieved unless powerful computational resources are available. Moreover, control design for such a complex system is generally infeasible. Hence, model order reduction should be applied.
We reduce the dimension of the full-order model (1) by using the RB method [2] and denote the RB solution obtained by using N RB functions with $\hat{\boldsymbol{U}}_N^n$ (similarly $\hat{y}_N^n$). We are interested in the computation of reliable error estimates and denote the difference between the FV and RB solution by $\boldsymbol{e}^n:=\boldsymbol{U}^n-\hat{\boldsymbol{U}}_N^n$ (similarly $e_y^n=y^n-\hat{y}_N^n$).

Error estimates for the reduced model

Following the idea introduced in [3] for linear systems and assuming L_{lin} in Σ_{lin} being a Schur matrix, an error bound on the ℓ_2-norm of the error signal is constructed as follows:

$$\|\boldsymbol{e_y}\|_{\ell_2}\le\gamma^{\boldsymbol{e_y}\mathcal{R}}\|\mathcal{R}\|_{\ell_2}+\gamma^{\boldsymbol{e_y}\boldsymbol{e_{U_{nl}}}}\|\boldsymbol{e_{U_{nl}}}\|_{\ell_2}, \tag{3}$$

where $\mathcal{R}^n:=\hat{\boldsymbol{U}}_N^{n+1}-\left(L_{lin}\hat{\boldsymbol{U}}_N^n+BU_0^n-\frac{\Delta t}{4\Delta x}L_{nl}\hat{\boldsymbol{U}}_{nl}^n+\frac{\Delta t}{2\Delta x}B(U_0^n)^2\right)$ is the residual obtained after reduction, $e_{\boldsymbol{U}_{nl}}$ is the approximation error of the nonlinear term $\boldsymbol{U}_{nl}$, and $\|\mathcal{R}\|_{\ell_2}:=\sqrt{\sum_{n=0}^{\infty}\|\mathcal{R}^n\|^2}$ (similarly for $\|\boldsymbol{e_y}\|_{\ell_2}$ and $\|\boldsymbol{e_{U_{nl}}}\|_{\ell_2}$).



(a) (b)

Figure 1: (a) Maximum error in the discrete parameter domain during the offline phase, (b) Error evolution by increasing the number of RB functions in the online phase.

Moreover, $\gamma^{\boldsymbol{e_y}\mathcal{R}}$ represents the ℓ_2-norm of the dynamical error system (obtained after subtracting the lifted reduced-order model from system (1)) from input $\mathcal{R}$ to the output $\boldsymbol{e_y}$ (similarly for $\gamma^{\boldsymbol{e_y}\boldsymbol{e_{U_{nl}}}}$). This ℓ_2-norm is equal to the $\mathcal{H}_\infty$-norm of the linear part of the dynamical error system with respect to the same input and output, which is computed as described in [3].

A second error estimate is defined following the idea presented by [4]. Assume that we have the reduced solution with two different levels of accuracy, one using N RB functions, the other one with $N'(>N)$ RB functions and we are interested in the error analysis for the case with N RB functions. We can relate the error of the two reduced models as follows:

$$\|y - \hat{y}_{N'}\|_{\ell_2} \leq \eta_N^{N'} \, \|y - \hat{y}_N\|_{\ell_2} \, . \tag{4}$$

Here, y is the actual output computed from (1) and $\hat{y}_N$ is obtained from the reduced model with N RB functions. In the so-called offline phase, N' is increased until the empirically obtained $\eta_N^{N'}$ becomes smaller than 1 for all parameters whose corresponding full-solution is available. Therefore, for any N, we can find N' such that $\eta_N^{N'} < 1$. This condition bears similarities with the small-gain condition mentioned in [3]. Now, in the offline phase, corresponding to each N, the value of N' and the value of $\eta_N^{N'}$ are known.

In the so-called online phase, two reduced solutions with N and N' RB functions should be solved. After obtaining these two computationally cheap solutions, we have:

$$\left.\begin{array}{r} \zeta_N^{N'} = \|\hat{y}_{N'} - \hat{y}_N\|_{\ell_2} \\ \|y - \hat{y}_N\|_{\ell_2} \leq \|y - \hat{y}_{N'}\|_{\ell_2} + \|\hat{y}_{N'} - \hat{y}_N\|_{\ell_2} \end{array}\right\} \xrightarrow{(4)} \|y - \hat{y}_N\|_{\ell_2} \leq \frac{\zeta_N^{N'}}{1 - \eta_N^{N'}} . \tag{5}$$

The reason for having the empirical factor $\eta_N^{N'} < 1$ shows itself here to have finite and positive error estimate.

The evolution of the error estimates and the actual error during the offline and online phase is shown in Figure 1. In general, the ℓ_2-gain estimate is sharper than the empirical one, but the latter one is faster with a larger applicability region.

Conclusions

In this work, a new error estimate based on the Lur'e type formulation of the nonlinear Burgers' equation is defined. This estimate is rigorous, accurate and effective, but has limited applicability. To circumvent this issue, hinged on the snapshots generated in the offline phase, an empirical error estimate is introduced. Both error estimates work efficiently in terms of computational effort and accuracy. However, the empirical error estimate is faster and also applicable on a wider range of problems compared to the error estimate proposed on the basis of ℓ_2-gain notion.

References

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