Nonsmooth dynamics of slip and stick with a finite-sized contact area

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Summary. Dry friction laws like the Coulomb law and many of its extensions predict forces that are discontinuous functions of slip velocity. In state space, this discontinuity occurs along discontinuity manifolds corresponding to sticking of the physical contacts. Previous works studied the dynamics induced by these models when the discontinuity manifold is codimension-one (Coulomb law in 2 dimensions) or codimension-two (Coulomb friction in 3 dimensions). Here we investigate the dynamics in more general contact models, which leads to higher codimension discontinuity manifolds. In particular, we analyse in details the Coulomb-Contensou friction model describing the friction forces and torques at a finite-sized contact area. Among others we show that the direction of slip velocity as well as the ratio between the spinning angular velocity and the slip velocity have predictable values at each transition between slip and stick.

Dynamics induced by discontinuous friction models

If completely rigid bodies are assumed with a single-point contact, the Coulomb friction law describes the friction force as a nonsmooth function of slip velocity. Namely,

$$\vec{\lambda}_f = -\mu\lambda_n \frac{\vec{s}}{|\vec{s}|}, \tag{1}$$

where $\lambda_n > 0$ is the normal force between the surfaces, $\vec{\lambda}_f$ is the tangential (friction) force, $\vec{s}$ is the relative velocity between the surfaces at the contact point, and μ is the friction coefficient. This model involves a discontinuity when $|\vec{s}| = 0$. In the case of planar (two-dimensional) friction problems with Coulomb friction, the discontinuity occurs along a codimension-1 manifold in phase space [1]. Such systems belong to the class of Filippov systems, for which a well-established theory describes when and how systems converge to discontinuity manifold and how they continue to evolve after that point.

When a planar model involves several point contacts, each of them induces a codimension-1 disconinuity manifold. These manifolds may intersect giving rise to secondary discontinuity manifolds of higher codimension, and a hierarchical system of discontinuities. The theory of Filippov systems was extended to such systems in [2] and [3].

A qualitatively different scenario occurs in the case of spatial (three-dimensional) friction problems. In that case, the Coulomb law gives rise to a discontinuity along an isolated codimension-2 manifold where both components of the slip velocity $\vec{s}$ vanish simultaneously. This lead to the recent introduction of the concept of *extended Filippov systems*, and the development of theory describing how systems behave in the neighbourhood of such discontinuities [4]. Among others, special features of slip-stick transitions, and the conditions of persistent stick motion were developed.

The Coulomb friction law requires refinement and extension where the stiffness of the contacting bodies is not large enough. Then, the local normal deformations have to be considered which creates a finite contact area. The Coulomb friction can be applied *locally* between the tangential and normal force distributions. By integration over the contact area, the resultant friction force $\vec{\lambda}_f$ and the friction torque τ_f can be computed with respect to a reference point of the contact area. This calculation can be found in the literature by using appropriate series expansion and closed form analytical approximations [5, 6]. It was found that the relative *slipping velocity* $\vec{s}$ and the *spinning angular velocity* ω are coupled in the tangential force $\vec{\lambda}_f$ and the drilling torque τ_f. According to those result in the literature, a simple phenomenological approximation of the friction law can be given as

$$\vec{\lambda}_f = -\mu_\lambda\lambda_n \frac{\vec{s}}{\sqrt{|\vec{s}|^2 + (c_\omega\omega)^2}}, \tag{2}$$

$$\tau_f = -\mu_\tau\lambda_n \frac{c_\omega\omega}{\sqrt{|\vec{s}|^2 + (c_\omega\omega)^2}}. \tag{3}$$

In (2)-(3), μ_λ, μ_τ are dimensionless coefficients proportional to the friction coefficient μ, and the model parameter c_ω characterises the interaction between the translational and rotational friction effects. As a limit case for very large local stiffness of the bodies, the contact area becomes a contact point, and these parameters are assumed to tend to $\mu_\lambda \to \mu$, $\mu_\tau \to 0$ and $c_\omega \to 0$. In this limit case, (2) leads to the Coulomb friction law (1) with a codimension-2 discontinuity. However, for a finite contact stiffness, the model has a discontinuity when $|\vec{s}| = 0$ and $\omega = 0$ at the same time. It means a codimension-3 discontinuity, but friction models including coupling between rolling and slipping (see e.g. [7]) are expected to initiate discontinuity manifolds up to the codimension-5 case. Currently there is no theory describing this class of non-smooth dynamical systems.

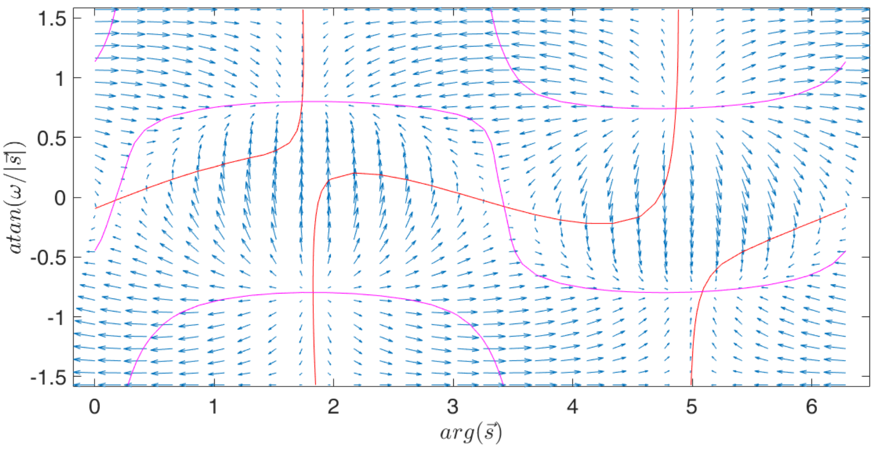

Figure 1: A projection of the 2-sphere with an example of the vector field representing fast dynamics of a slipping contact in the presence of the friction model (2)-(3). The curves represent nullclines of the dynamics. The field has 6 fixed points at the intersections of the nullclines.

Vector fields with higher codimension discontinuities

The recent theory of *extended Filippov systems* [4] addresses dynamics of a system with state variables $\vec{x}$ in the neighborhood of a codimension-2 discontinuity manifold of state space that occurs at $\vec{u}(\vec{x}) = 0$ for some $\vec{u}(\vec{x}) \in \mathbb{R}^2$. The theory is based on the observation, that the variable $\vec{u}$ can be replaced by polar variables $r = |\vec{u}|$, and $\phi = \arg(\vec{u})$ (i.e. the angle of $\vec{u}$). Then, for $r << 1$, ϕ evolves on a faster time-scale than r. The theory of smooth slow-fast dynamical systems offers an efficient tool to analyze the emerging dynamics. In most cases, ϕ rapidly converges to fixed points, and the slow dynamics of r is evaluated at those points.
Here, we apply the same technique to systems where a codimension n discontinuity occurs at $\vec{u} = 0$, $\vec{u} \in \mathbb{R}^n$. We again decompose $\vec{u}$ to a slow variable $r \in \mathbb{R}$ and fast variables $\vec{\phi} \in \mathbb{S}^{n-1}$ over the $n-1$ sphere. In many cases, the fast dynamics converges to fixed points, where the one-dimensional slow dynamics can be analyzed easily.

Application to the dynamics of finite-sized contacts

Consider a rigid multi-body system with smooth behavior except for one single finite-sized contact area with dry friction. The state variable $\vec{x}$ consists of the generalized coordinates $\vec{q}$ and their time derivatives $\dot{\vec{q}}$. The equations of motion and kinematic constraints enable one to develop equations for the dynamics of the slip velocity $\vec{s}$ and angular velocity of slip ω in the following form:

$$\dot{\vec{u}} := \begin{bmatrix} \dot{\vec{s}} \\ \dot{\omega} \end{bmatrix} = \mathbf{M}^{-1}(q) \begin{bmatrix} \vec{\lambda}_f \\ \tau_f \end{bmatrix} + \vec{b}(q, \dot{q}) \tag{4}$$

Here $\vec{b}$ is the acceleration in the absence of frictional forces and $\mathbf{M}$ is a 'local mass matrix' associated with the reference point of the contact region. One can combine this equation with a friction model like (2)-(3). Then, replacement of $\vec{u}$ by the variables r and $\vec{\phi}$ as described above uncovers intriguing fast dynamics over the 2-sphere when $r << 1$ (Fig. 1). It appears that the limit sets of the fast dynamics are two to six fixed points. The slow dynamics is either convergence to or divergence from $r = 0$ (i.e. stick) at each of the fixed points. This observation implies that slip-stick and stick-slip transitions always occur through non-trivial combinations of rotational and translational motion at the contact surface. The analysis also hints at the possibility of ambiguous cases where persistence of stick is undecidable within the scope of rigid body theory.

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