Nonlinear Dynamics of a Shearable-Extensible Beam with an Elastic Longitudinal Support: Analytical Derivation, Numerical Simulation and Experimental Validation

Lukasz Kloda*†, Stefano Lenci† and Jerzy Warminski*
*Department of Applied Mechanics, Lublin University of Technology, Lublin, Poland
†Department of Civil and Buildings Engineering and Architecture, Polytechnic University of Marche, Ancona, Italy

Summary. Coupled axial-transversal nonlinear oscillations of a simply supported beam with an axial spring are studied in the paper. The exact model of a planar beam with associated boundary conditions is derived, and then to analyze free and forced-damped dynamics of the structure the perturbation method up to cubic nonlinearity is used. Next, a finite element model of the beam-spring system is considered and then outcomes are compared. Experimental tests on a slender beam confirmed quantitatively analytical and numerical models.

Introduction

In the *classical beam theory* for vibrations only transverse inertia forces and bending moment are considered [1], and the shearing effect together with longitudinal deformations are neglected. Those assumptions are relatively correct for slender structures with axially restrained ends. Large amplitudes of a thick beam require the Timoshenko's shearing theory, and for the beam unrestrained in axial direction longitudinal inertia terms can not be neglected [2]. The inertia changes the dynamics of the structure, and as a consequence the simply supported beam has no longer a hardening nature but a softening behaviour. An axial spring subjected to the unrestrained end allows to passively control hardening/softening dichotomy [3, 4]. The goal of this paper is to investigate nonlinear dynamics of the beam-spring system and its sensitivity on boundary conditions by considering nonlinear coupling between transversal and longitudinal modes. For model validation a laboratory test is prepared.

Beam models

Analytic

In the paper two beam models are used, both of them assume linear elastic material properties as Young (E) and shear (G) modules (Eqs. (1)). It means that nonlinearities arise only from geometrical and inertia coupling. In the reference (rest) configuration the beam has length L and rectangular cross-section A. Normal, shearing and bending forces are proportional to elongation $\hat{e}$, shear strain γ and curvature k_g, respectively:

$$N = EA\hat{e}, \quad V = GA\gamma, \quad M = EJk_g. \tag{1}$$

Note that the beam is *extensible* . W is the longitudinal displacement, and U is the transversal one. The *geometrical* definition of curvature [5] is adopted in our study, see Eq. (2). θ describes the rotation of the beam's cross-section (Fig. 1a) which contains the slope angle φ and γ. The deformations are given by

$$\hat{e} = S' - 1, \quad S' = \sqrt{(1+W')^2 + U'^2}, \quad k_g = \frac{\theta'}{S'} = \frac{\theta'}{\sqrt{(1+W')^2 + U'^2}}, \quad \theta = \varphi + \gamma. \tag{2}$$

Decomposing strain forces into horizontal, vertical and rotational coordinates and supplementing them by inertia ($\rho A\ddot{W}$, $\rho A\ddot{U}$ and $\rho J\ddot{\theta}$), damping ($C_W\dot{W}$, $C_U\dot{U}$, $C_\theta\dot{\theta}$) and external forces (P_W, P_U, P_θ) the following balance equations are obtained

$$(N\cos\varphi + V\sin\varphi)' = \rho A\ddot{W} + C_W\dot{W} + P_W(Z,T), \tag{3}$$
$$(N\sin\varphi - V\cos\varphi)' = \rho A\ddot{U} + C_U\dot{U} + P_U(Z,T), \tag{4}$$
$$M' - VS' = \rho J\ddot{\theta} + C_\theta\dot{\theta} + P_\theta(Z,T), \tag{5}$$

where dots $\dot{()}$ are time derivative and primes $()'$ depict partial derivative with respect to the coordinate Z. The stiffness of the axial end spring k_s is responsible for change of boundary conditions only in the longitudinal direction:

$$U(0,T) = 0, \quad U(L,T) = 0, \quad M(0,T) = 0, \quad M(L,T) = 0, \quad W(0,T) = 0, \tag{6}$$
$$N(L,T)\cos\varphi + V(L,T)\sin\varphi + W(L,T)k_s = 0. \tag{7}$$

To solve the set of three equations (3)-(5) and associated boundary conditions (6)-(7) the multiple time scales method (*MTSM*) is applied, and after cumbersome computations, the frequency response curves are drawn for different $0 < k_s < \infty$ parameter [6].

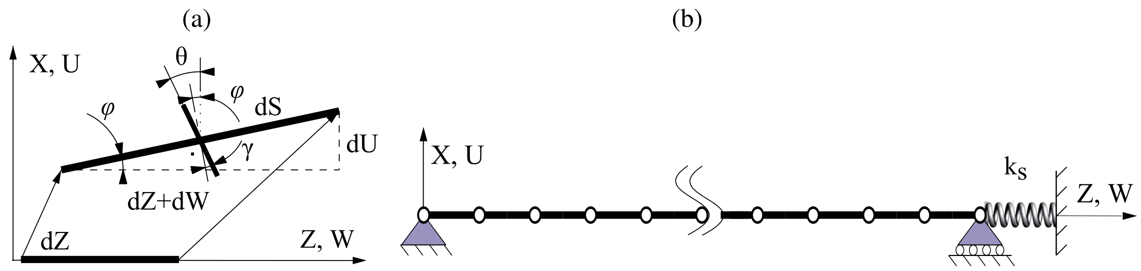

Figure 1: Kinematics of the beam segment in analytical approach (a) and finite element model of the hinged-simply supported beam (b).

Finite element

Numerical computations are made with the commercial software Abacus_CAE. Dynamical analysis is performed in two modules: *linear perturbation-frequency* and transient in time *dynamic explicit*. The finite element model (*FEM*) presented in Fig. 1(b) is composed of 100 segments *B21*-type plus one linear spring connector k_s. Boundary conditions are consistent with (6)-(7). Four different computational methods are applied: (i) linear modal analysis, (ii) free nonlinear dynamics, (iii) following-path and (iv) shooting method. All four methods complement each other and allow a deep study of any dynamical system for selected parameters to discover unexpected phenomena. However the computations in time domain are very time consuming.

Results and conclusions

Comparison of backbone curves obtained by *FEM* and *MTSM* is presented in Fig. 2(a). After passing the transient vibrations the simulation for steady states are in excellent agreement. For Timoshenko beam with partial tip reinforcement, the structure has hardening behaviour. Due to technical limitations experimental tests have been simplified to a kinematically excited slender beam presented in Fig. 2(c) and, then overlapped with numerical counterpart. The axially unrestrained case represents softening phenomenon and again results shown in Fig. 2(b) are in very good agreement. We conclude that analytical and numerical results are confirmed by experiments with excellent compliance.

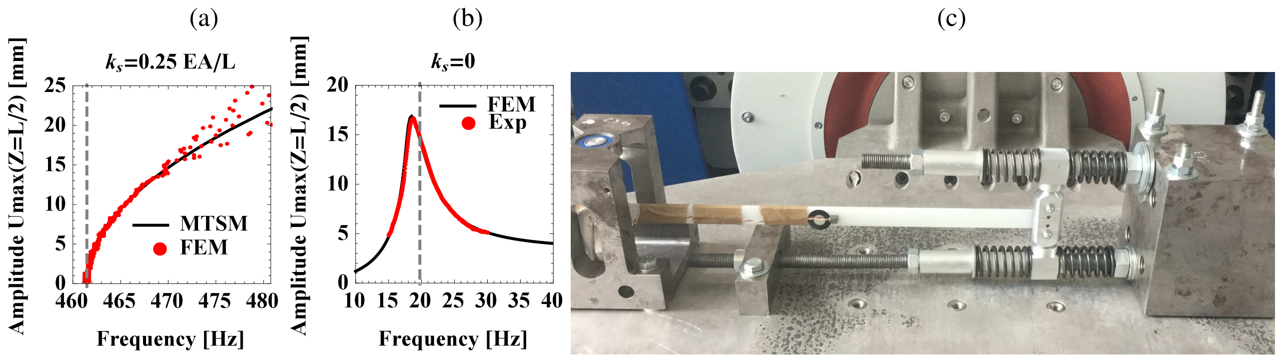

Figure 2: Free nonlinear oscillations (a), forced damped oscillations (b) and experimental setup - side view (c).

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