An Efficient Implementation for the Analysis of Extrema in Dynamical Systems with Delay

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Summary. We provide a general framework for optimization along periodic orbits or quasiperiodic invariant tori in dynamical systems with delay. Our recent study [1] developed a methodology for such problems based on the calculus of variations. The formulation presented in [1] did not exploit the typical structure of Jacobians in problems with delay, resulting in high computational costs and difficult-to-generalize algorithms. Here, we reformulate general boundary-value problems for delay equations by decomposing them into ordinary differential equation and algebraic interval coupling conditions. We consider such coupling conditions for various multi-segment boundary-value problems and describe how the necessary optimality conditions along with the corresponding Jacobians may be implemented in the continuation package COCO [2]. Several examples demonstrate the implications to computational efficiency, as well as the ease of problem construction.

Motivating example

Consider the retarded delay-differential equation with periodic forcing

$$\dot{x} = f\left(t, x, g(x,p)(t), p\right) := \begin{pmatrix} -\omega x_2 + x_1\left(t-\alpha\right)\left(1 + r\left(\cos\left(2\pi t/T - 1\right)\right)\right) \\ -\omega x_1 + x_2\left(t-\alpha\right)\left(1 + r\left(\cos\left(2\pi t/T - 1\right)\right)\right) \end{pmatrix}, \tag{1}$$

where $r = \sqrt{x_1^2 + x_2^2}$ and $p = (\alpha, \omega, T)$. Figure 1 shows a family of two-dimensional quasiperiodic invariant tori for this system obtained for fixed delay α and rotation number ρ (ratio of fundamental frequencies). In Fig. 1, the fold point at $\omega \approx 0.44$ separates stable (solid) from unstable (dashed) tori. We may locate the fold and further seek to locate extrema in ω under additional variations in α by considering the set of necessary conditions for stationary values of ω given corresponding dynamic constraints. Specifically, by suitable transformations [1, 2], an equivalent representation for a quasiperiodic invariant torus of (1) on the cylindrical domain $(\varphi, \tau) \in \mathbb{S} \times [0, 1]$ (here, $\mathbb{S}$ denotes the unit circle) is in terms of a solution to the differential constraint

$$V_{,\tau}(\varphi, \tau) = Tf\left(T\tau, V(\varphi, \tau), \left[W_{\frac{\alpha}{T}}V\right](\varphi, \tau), p\right) \tag{2}$$

along with the boundary condition

$$V\left(\varphi, 1\right) - V\left(\varphi + 2\pi\rho, 0\right) = 0. \tag{3}$$

Here, the wrapping operator W_a is given by

$$\left[W_a\chi\right](\varphi, \tau) := \chi\left(\varphi - 2\pi j\rho, \tau - a + j\right), \tag{4}$$

where $j \in \mathbb{Z}$ is defined depending on a, φ, τ such that $\tau - a + j \in [0, 1)$. This operator captures the dependence of the vector field on a shifted V, consistent with the presence of delay. Let $f_{,2}(\varphi, \tau)$ and $f_{,\omega}(\varphi, \tau)$ denote the partial derivatives of f with respect to its second argument and ω, respectively, evaluated at $\left(T\tau, V(\varphi, \tau), \left[W_{\frac{\alpha}{T}}V\right](\varphi, \tau), p\right)$, let df/dT denote the total derivative of $f\left(T\tau, V(\varphi, \tau), \left[W_{\frac{\alpha}{T}}V\right](\varphi, \tau), p\right)$ with respect to T, and let $h_j(\varphi, \tau)$ denote the partial derivative of f with respect to its third argument evaluated at $\left(T\tau, V, V\left(\varphi - 2\pi j\rho, \tau - \frac{\alpha}{T} + j\right), p\right)$ for $j = 0, 1$. A set of suitably constructed necessary adjoint conditions for stationary values of ω are then given by

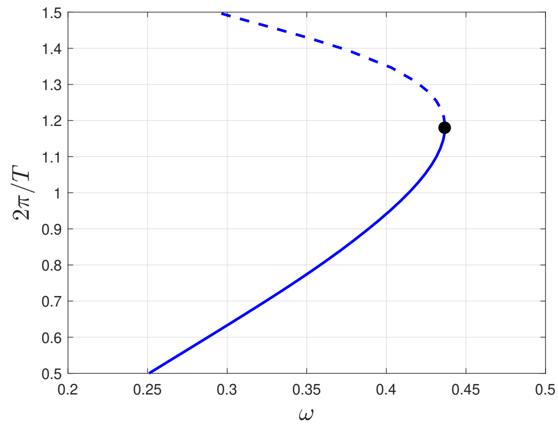

Figure 1: One-dimensional family of quasiperiodic tori [1] obtained with rotation number $\rho \approx 0.6618$ and $\alpha = 1$.

$$-\lambda_{f,\tau}^{\mathrm{T}} - T\lambda_f^{\mathrm{T}} f_{,2} - T\left(W_{-\frac{\alpha}{T}}\lambda_f\right)^{\mathrm{T}} W_{-\frac{\alpha}{T}} h_0 = 0,\ (\varphi, \tau) \in \mathbb{S} \times \left(0, 1 - \alpha/T\right), \tag{5}$$

$$-\lambda_{f,\tau}^{\mathrm{T}} - T\lambda_f^{\mathrm{T}} f_{,2} - T\left(W_{-\frac{\alpha}{T}}\lambda_f\right)^{\mathrm{T}} W_{-\frac{\alpha}{T}} h_1 = 0\ (\varphi, \tau) \in \mathbb{S} \times \left(1 - \alpha/T, 1\right), \tag{6}$$

and

$$\lambda_f^{\mathrm{T}}\left(\varphi, 0\right) + \lambda_{\mathrm{rot}}^{\mathrm{T}}\left(\varphi - 2\pi\rho\right) + \lambda_{\mathrm{ph}} V_{,\varphi}^{\star\mathrm{T}}\left(\varphi\right) = 0, \qquad \lambda_f^{\mathrm{T}}\left(\varphi, 1\right) + \lambda_{\mathrm{rot}}^{\mathrm{T}}\left(\varphi\right) = 0, \tag{7}$$

$$-\int_0^{2\pi}\int_0^1 \lambda_f^{\mathrm{T}} T f_{,\omega}\,\mathrm{d}\tau\,\mathrm{d}\varphi + \eta_\omega = 0, \qquad -\int_0^{2\pi}\int_0^1 \lambda_f^{\mathrm{T}}\left(f + T df/dT\right)\mathrm{d}\tau\,\mathrm{d}\varphi + \eta_T = 0 \tag{8}$$

in terms of the original variables V, ω, and T, a reference function $V^\star$, and a set of Lagrange multipliers λ_f, λ_{rot}, λ_{ph}, η_ω, and η_T, where $\eta_\omega = 1$ and $\eta_T = 0$ at a stationary point. The analysis in [1] demonstrates the successful location of such stationary points (and of the fold in Fig. 1) using a method of successive continuation, first pioneered by Kernévez and Doedel [3].

Software implementation

In the present work, we describe an effort to improve the computational efficiency of the implementation of the continuation problem in the previous section. Specifically, in contrast to the implementation in [1], which involved numerically computed Jacobians and no vectorization of the corresponding discretized problem, we describe an ongoing effort to encode explicit Jacobians in a vectorized form that supports an entire class of continuation problems with delay. These include initial-value problems, multisegment periodic orbits, and quasiperiodic invariant tori, even in the presence of multiple delays. The development platform is COCO, a MATLAB-based software package with an object-oriented construction paradigm that supports building composite problems by coupling instances of simpler problems. As an example, continuation of quasiperiodic invariant tori for a problem without delay was considered in [2] in terms of a multisegment boundary-value problem with all-to-all coupling of segment end points in terms of a suitably formulated Fourier interpolant. This past works motivates the implementation sought in the presence of delay.

Problem decomposition

To illustrate our approach, consider replacing the equation

$$x' = f(\tau, x, g(x,p)(\tau), p) \tag{9}$$

with the decomposition

$$x' = f\left(\tau, x, y, p\right), \, y\left(\tau\right) = g\left(x, p\right)\left(\tau\right) \tag{10}$$

for $\tau \in (0,1)$ in terms of a differential equation and an interval condition. A special case is obtained for $g(x,p)(\tau) := x(\tau - p_1|_{\mathrm{mod}\,[0,1)})$ corresponding to a periodic-orbit problem (omitting the discrete condition of periodicity at $\tau = 0$) with a single discrete delay p_1. In this case, vanishing variations of the Lagrangian

$$L = \int_0^{p_1} \lambda_{\mathrm{int}}^\top(\tau)\left(y(\tau) - x(1+\tau-p_1)\right)\,\mathrm{d}\tau + \int_{p_1}^{1} \lambda_{\mathrm{int}}^\top(\tau)\left(y(\tau) - x(\tau-p_1)\right)\,\mathrm{d}\tau \tag{11}$$

with respect to λ_{int} yield the original interval condition, while variations with respect to x yield the contributions $-\lambda_{\mathrm{int}}^\top(\tau + p_1 - 1)$ and $-\lambda_{\mathrm{int}}^\top(\tau + p_1)$ on $\tau \in (1-p_1, 1)$ and $\tau \in (0, 1-p_1)$, respectively, to the adjoint differential equations. In general, the proposed decomposition allows us to develop a fully vectorized encoding with explicit Jacobians of a discretization of the differential equation constraint in (10) in terms of a corresponding discretization of x and y, without simultaneously imposing (a discretization of) the interval coupling constraint in (10). It is clear that we may similarly derive a general form of the discretized contributions to the adjoint conditions associated with the differential equation constraint, as well as their Jacobians, without also considering the contributions from the interval coupling constraint. Once these have been encoded in all generality, they may be invoked any number of times in the construction of a multisegment trajectory problem, such as that associated with continuation of quasiperiodic invariant tori. Of course, in a multisegment trajectory problem, the interval coupling conditions typically take a more general form in which y's associated with different segments are expressed in terms of x's associated with multiple other segments. A challenge is the derivation of a general form of the corresponding adjoint conditions, their discretization, and corresponding Jacobians.

Conclusions

A decomposition of the implementation of the necessary conditions for stationary values of an objective function along a constraint manifold defined by a boundary-value problem with delay is proposed in order to support optimization along families of periodic orbits or quasiperiodic invariant tori. The decomposition separates differential constraints from a set of interval conditions by the introduction of additional auxiliary unknowns. The advantage of the proposed decomposition is in the convenience of the implementation, including the vectorization of the discretized constraints and their Jacobians, and its generalization to a number of other problem types, including differential-algebraic problems. We argue that the expense of increasing the number of unknowns is outweighed by the shortened development time for different classes of user-specific optimization problems, per the underlying philosophy of the COCO software development.

References

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