# Experimental bifurcation analysis of a self-excited system exhibiting a subcritical Hopf bifurcation using control-based continuation

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*Summary*. In this paper, control based continuation (CBC) is exploited to systematically characterise in a controlled manner the LCOs of an airfoil during wind tunnel tests. Limit cycle oscillations (LCOs) are found in a wide range of engineering systems such as aircraft, valves, towed wheels and machine tools. The development of corresponding mathematical models that can accurately predict the region where LCOs exist is challenging, especially when the LCOs coexist with the stable equilibrium of the system. Experimental validation is also challenging as the LCOs that can be observed during tests can have large amplitudes. Contrary to previous applications of CBC, the present system is autonomous and the frequency of oscillation is therefore a priori unknown. A phase-plane (geometric) control approach is used to overcome this difficulty. Experimentally measured bifurcation diagrams are then exploited for parameter estimation and model validation.

## Introduction

Limit cycle oscillations (LCOs) are found in a wide range of engineering systems such as aircraft, valves, towed wheels and machine tools. In those systems, LCOs emerge from a Hopf bifurcation which can either be super- or sub-critical. The latter is particularly problematic as the stable equilibrium of the system will coexist with LCOs that have potentially large amplitudes. A transition from the equilibrium to such a large-amplitude LCO could cause unacceptable oscillation amplitudes and even lead to catastrophic failures.

The development of mathematical models that can accurately capture the region where the equilibrium and the LCOs coexist is key to the proper operation and safety of those engineering systems. However, this is very difficult task as it often entails capturing complex nonlinear physical phenomena such as fluid-structure interactions (see example here below). Furthermore, parameter estimation and model validation may not be able to exploit LCO data as it would be unsafe to operate the system in this regime of motion.

In this paper, we overcome these difficulties using control-based continuation (CBC). CBC is an experimental scheme that combines feedback control with path following techniques to systematically investigate the dynamic behaviour of physical systems. CBC was applied to a wide range of mechanical experiments such as a impact oscillators, nonlinear energy harvesters, and a cantilever beam with a nonlinear mechanism at its free tip [3, 2, 4]. Here, CBC is exploited to characterise the dynamics of a two-degree-of-freedom airfoil exhibiting LCOs through a subcritical Hopf bifurcation. Testing such a system with a feedback controller provides numerous advantages compared to classical open-loop experiments. If properly designed, the controller maintains the response of the system around a prescribed operating point, avoiding untimely transitions to other, potentially dangerous behaviours. The controller also modifies the linearization of the dynamical system such that unstable responses are stable and hence observable. The unstable LCOs correspond to smaller oscillation amplitudes which can be safely measured and exploited for parameter estimation and model validation.

## Application of CBC to LCOs

The fundamental idea of CBC is to find a suitable target signal for the controller such that it becomes *non-invasive* and does not modify the position in parameter space of the responses of the open-loop experiment of interest. The target signal is usually described by a finite sum of Fourier modes whose coefficients can be found iteratively. Until now, the frequency of the target signal was known a priori and identical to the frequency of the external excitation applied to the system (see, for instance, [2]). This is no longer the case here where the system is autonomous.

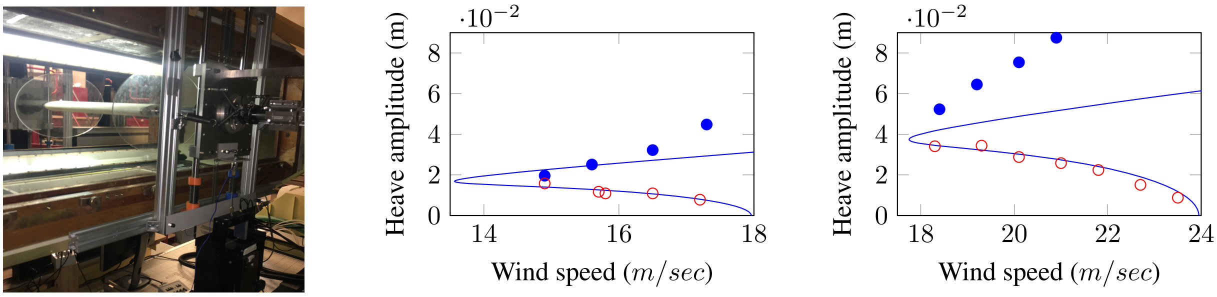
To overcome this difficulty and obtain the appropriate frequency for the target signal, the fundamental component of the target signal $z^*(t)$ is phase locked with the measured response of the system $z_1(t)$. This is equivalent to locking the phase between force and the response of the forced Hopf bifurcation. For simplicity, the target signal is also limited to a single Fourier mode. Assuming that the measured signal $z_1$ is analytic, it and it's time derivative can be expressed as

$$\begin{aligned} z_1(t) =& A(t)\cos(\phi), \\ z_2(t) =& \dot{z}_1(t) = -\dot{\phi}A(t)\sin(\phi) \end{aligned} \tag{1}$$

where, $A(t)$ is the instantaneous amplitude, $\phi$ is the instantaneous phase of the response, $\dot{\phi}$ is the frequency of the response. The target signal can then be set as

$$z^*(t) = \hat{A}\cos(\phi) \qquad \text{where} \qquad \phi = \tan^{-1}\left(\frac{-z_2(t)}{\dot{\phi}z_1(t)}\right) \tag{2}$$

and $\hat{A}$ is the unknown amplitude of the control target. The amplitude coefficient $\hat{A}$ is found such that the controller becomes non-invasive. This is achieved by solving the zero problem $\Xi = \hat{A} - A(\hat{A})$ where $A$ is amplitude of the measured




(a) (b) (c)

Figure 1: Picture of the flutter rig and CBC results of two systems. (a) Picture of the rig (b) CBC result and numerical computation of LCO of the model (system 1) (c) CBC result and numerical computation of LCO of the model (system 2). (●) is measured stable LCO, (○) is measured unstable LCO, (——) is computed LCO from the model.

response. This one-dimensional problem can be solved using Newton-like algorithms or, more simply, by testing a range of target amplitudes and taking the one for which $|\Xi_k| < \delta$, where $\delta$ is a user-defined tolerance.

## Experimental results

CBC was applied to a flutter rig (Figure 1(a)) with a rigid NACA-0015 wing profile and two degrees of freedom (heave and pitch). The heave motion was measured using a laser displacement sensor and the control force was applied to the heave motion using an APS Electro-Seis Shaker. The CBC algorithm were run on a laptop computer connected to a real-time controller (RTC). The RTC consists of a BeagleBone Black on which is fitted with a custom data acquisition board [1] in which the real-time feedback controller was implemented. Measurement and actuation was performed with a sampling frequency of 5 kHz.
The bifurcation diagram obtained using CBC is presented in Figure 1 (b)-(c). LCOs measured at different wind velocities represent either unstable (○) or stable (●) LCOs of the underlying uncontrolled experiment. The location of the sub-critical Hopf bifurcation and the LCOs represent valuable information that was exploited to estimate the nonlinear parameters of the mathematical model. This models accounts for the presence of linear and nonlinear stiffnesses in heave and pitch. It also includes an unsteady representation of the aerodynamics. The bifurcation diagrams found for this model are shown in Figure 1(b)-(c) for two diffrent configurations. The agreement between the model and the unstable responses is very good. The significant difference observed at high amplitudes between the model and the experimentally-measured stable LCOs is thought to be due to complex aerodynamic phenomena that are not included in the model.

## Conclusions

Control-based continuation was exploited to characterise the limit cycle oscillations of a flutter rig in wind tunnel tests. The control target signal was phase-locked with the response of the system to find and match the frequency of oscillation. Identified LCOs were exploited to estimate parameters of a model and reproduce numerically the bifurcation diagram of the system.

## Acknowledgements

KHL is funded by a University of Bristol Scholarship, D.A.W.B. is funded by the EPSRC grant EP/K032738/1 and LR is funded by a Research Fellowship from the Royal Academy of Engineering (RF1516/15/11). The University of Bristol, the EPSRC and the Royal Academy of Engineering are gratefully acknowledged.

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