Tangencies in the phase space of mechanical systems with spatial Coulomb friction

Mate Antali
Department of Applied Mechanics, Budapest University of Technology and Ec., Budapest, Hungary

Summary. When modelling dry friction between rigid bodies, the Coulomb friction law leads to discontinuity surfaces in the phase space of the resulting dynamical system. This discontinuity is a codimension-1 subspace in the case of a planar (two-dimensional) contact and it is a codimension-2 subspace in the case of a spatial (three-dimensional) contact. The qualitative mathematical description of the trajectories is well-established in the literature for the former case, and some results has been recently added to the latter topic by the author. In this work, the tangency points are analysed, which points divide the discontinuity surface to 'sliding regions' and 'crossing regions'. These regions coincide to the existence or non-existence of rolling-sticking solutions, thus, tangencies as boundary points are strongly related to these physical phenomena. The qualitative analysis of tangencies at codimension-2 discontinuities are carried out by analytical methods and the results are demonstrated on mechanical problems with spatial Coulomb friction.

Tangencies in piecewise smooth vector fields

The mathematical concepts are presented in the simplest geometrical case in the phase space, but the formulation can easily modified for more general situations. Consider a dynamical system in the form $\dot{x} = F(x)$, where $x = (x_1, \dots x_m) \in \mathbb{R}^m$. We assume that the vector field F is smooth on $\mathbb{R}^m$ except in a *discontinuity set* $\Sigma = \{x : x_1 = 0\}$. We call F a *piecewise smooth* vector field according to the two smooth regions $x_1 > 0$ and $x_1 < 0$. The set Σ has $m-1$ dimensions, thus, we call it a codimension-1 discontinuity. Consider the normal vector $n = (1, 0, \dots 0)$ of Σ at a chosen point $\tilde{x} \in \Sigma$. If the limits

$$\lim_{\varepsilon \to 0^+} F(\tilde{x} + \varepsilon n) = F^+(\tilde{x}), \qquad \lim_{\varepsilon \to 0^+} F(\tilde{x} - \varepsilon n) = F^-(\tilde{x}), \tag{1}$$

exist and $F^+(\tilde{x}) \neq F^-(\tilde{x})$ for all $\tilde{x} \in \Sigma$ then F is called a *Filippov-system*. In the vicinity of the discontinuity set, two generic types of behaviour occur according to the first components F_1^+ and F_1^- of the limit vectors. The case $F_1^+ \cdot F_1^- > 0$ is called *crossing*, where the trajectories of F can be continued through the discontinuity. The case $F_1^+ \cdot F_1^- < 0$ is called *sliding*. In this case, the so-called *sliding dynamics* can be defined inside Σ by using the convex set generated by F^+ and F^- (for the details, see [1] or [2]).

The boundary between these two cases is when $F_1^+ = 0$ or $F_1^- = 0$, which is called a *tangency point* or simply a *tangency*, because the vector field is tangent to the discontinuity set from either side. Tangencies separate the crossing and sliding regions of the discontinuity set. Tangencies often considered as singularities of the phase space; the analysis of these points gives qualitative information about the local behaviour of the vector field. The simplest tangencies are called *invisible fold* and *visible fold* (see Figure 1), more complicated cases can be found in the literature (see [3] and Chapter 6 of [4]).

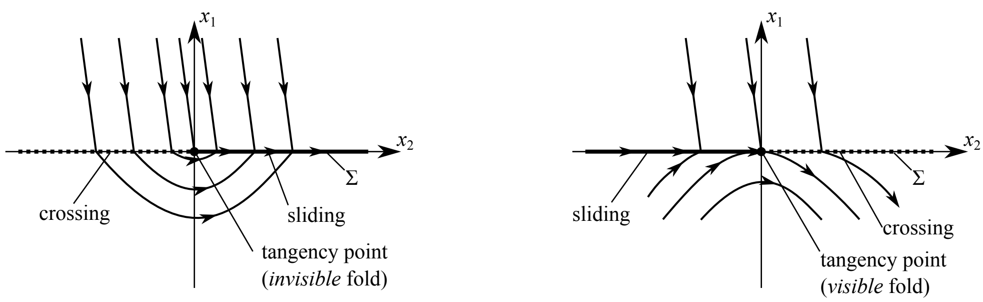

Figure 1: Two basic types of tangencies in piecewise smooth systems. Left panel: invisible fold. The term 'invisible' expresses that although the vector field is tangent to the discontinuity set Σ, we can find no finite trajectory which is tangent to Σ. Right panel: visible fold. The term 'visible' expresses that there exists a trajectory which just touches Σ at the tangency point.

Analysis of tangencies at codimension-2 discontinuities

Let us now turn to the systems with codimension-2 discontinuities. Consider a system $\dot{x} = F(x)$ in the case when F is smooth everwhere except in the set $\Sigma = \{x : x_1 = x_2 = 0\}$. That is, Σ is now a codimension-2 discontinuity. Then, the continuously many normal directions to Σ can be parametrised by an angle ϕ as $n(\phi) = (\cos\phi, \sin\phi, 0, \dots 0)$. We assume that for all $\tilde{x} \in \Sigma$, the limit

$$\lim_{\varepsilon \to 0^+} F(\tilde{x} + \varepsilon n(\phi)) = F^*(\tilde{x}, \phi) \tag{2}$$

exist for all angles $\phi \in [0, 2\pi)$. Then, we can call the vector field as an extended Filippov system. In such systems, the sliding and crossing regions can be defined by transforming to polar coordinates and using the concept of limit directions

[5]. We can define the concept of tangency points directly by requiring that the vector field is *tangent* to the discontinuity if both components F_1^* and F_2^* of the limit vector vanish in the normal plane to Σ. That is, we call $\tilde{x}$ a tangency point if there exists a direction $\phi_1 \in [0, 2\pi]$ such that $F_1^*(\tilde{x}, \phi_1) = 0$ and $F_2^*(\tilde{x}, \phi_1) = 0$.

In this work, the tangencies can be analysed by similar tools those of the tangencies of codimension-1 discontinuities. However, these methods should be adjusted to the specialities of the codimension-2 discontinuity set. The vector field can be transformed to a truncated series form and by analytical tools, the local structure of the trajectories can be determined. The singularity conditions of the different types of the tangencies are investigated. Moreover, regularization is applied at the discontinuity, which provides a further insight into the problem.

Tangencies and mechanical problems with Coulomb friction

When Coulomb friction is assumed between rigid bodies, the two-dimensional contact problem usually leads to a piecewise smooth system where $x_1 = u$ is the relative velocity at the contact point. Then, the static (*sticking* or *rolling*) contact state corresponds to the sliding dynamics inside the discontinuity set while the *slipping* state occurs outside the discontinuity. The sliding region coincides with the region where the rolling/sticking state is permitted by the friction model. Similarly, the crossing region coincides to the region where the sticking state is not available due to slipping. It can be shown [6] that in the case of spatial Coulomb friction with a codimension-2 discontinuity, the same coincidence appears between these mathematical objects of the phase space and the mechanical phenomena.

In both the planar and spatial cases, the tangencies are located at the boundary where the rolling/sticking motion becomes realizable. Therefore, the local analysis of the vector field at the tangencies can be applied to explore the behaviour of bodies at the limit of slipping.

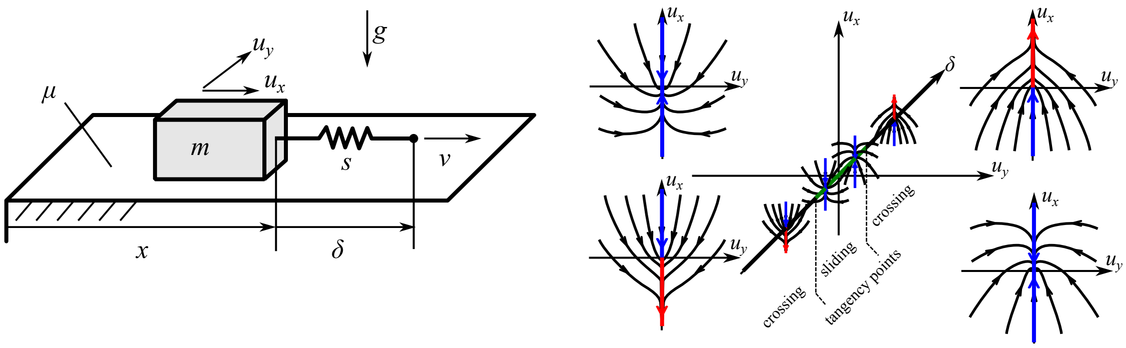

Figure 2: A three-dimensional mechanical model containing tangencies as important singularities at the boundary of slipping and sticking. Left panel: the sketch of the model. A block is pulled on a rough surface with a spring. The mass of the block is m, the stiffness of the spring is s, the friction coefficient on the surface is μ, and the end of the spring is pulled with a constant velocity v. The state space of the block can be described by $x = (u_x, u_y, \delta)$ where u_x and u_y are the components of the slipping velocity of the block and δ denotes the deformation of the spring. Right: the sketch of the phase space of the system with the projection of some typical trajectories onto the normal plane of the discontinuity. The blue and red half-lines denote the attracting and repelling limit directions, respectively.

The results are demonstrated on mechanical examples. For example, consider Figure 2, where we can see a block pulled by a spring on a rough surface. Analysis of the phase phase shows that there are two tangency points and one of them are a visible fold, the another is an invisible fold. It can be shown that although trajectories can reach the sliding region of the discontinuity, the only way to leave it is going through the *visible fold tangency*, that is, any solution containing sticking must go through the tangency point. In more complicated problems with a higher dimension of the phase space, the tangency point form boundary surfaces in the discontinuity. These surfaces behave as separatrices between the qualitatively different branches of solutions.

References

[1] Di Bernardo M. et al. (2008) Piecewise-smooth Dynamical Systems. Springer, London.

[2] Leine R., Nijmeijer H. (2004) Dynamics and Bifurcations of Non-Smooth Mechanical Systems. Springer, Berlin Heidelberg.

[3] Kristiansen K.U., Hogan S.J. (2019) On the interpretation of the piecewise smooth visible-invisible two-fold singularity in $\mathbb{R}^3$ using regularization and blowup *Journal of Nonlinear Science* **29**(2):723-787.

[4] Jeffrey M.R. (2019) Hidden Dynamics – The Mathematics of Switches, Decisions and Other Discontinuous Behaviour. Springer Nature, Switzerland.

[5] Antali M., Stepan G. (2018) Sliding and crossing dynamics in extended Filippov systems *J. Appl. Dyn. Syst.* **17**(1):823-858.

[6] Antali M., Stepan G. (2019) Nonsmooth analysis of three-dimensional slipping and rolling in the presence of dry friction *Nonlinear Dynamics* **97**(3):1799-1817.