Control-based continuation of orbits with complex time profile

Jan Sieber*, Courtney Quinn†
**Department of Mathematics, University of Exeter, UK*
†CSIRO, Hobart, Austrialia

Summary. We illustrate how unstable trajectories with complex time profiles and large period can be tracked using feedback control-based continuation. The experimental or computational effort is proportional to the period. The approach requires the feedback control to be stabilizing within time of order 1 uniformly along the orbit. The approach is illustrated with a stochastic simulation of a delay model for the Mid-Pleistocene transition of the palaeoclimate ice ages.

Control-based continuation

Control-based continuation applies feedback control to turn a controllable nonlinear dynamical system with inputs and outputs into a system of nonlinear equations, which can then potentially be solved by general-purpose nonlinear solvers or continuation (curve tracking) algorithms; see [2, 6, 7] by Renson, Barton *et al* and Schilder *et al* for detailed descriptions of the methodology. The approach assumes that the user (e.g., experimenter) has implemented a stabilizing feedback control loop. One may assume that the dynamical system follows an ODE of the type

$$\dot{x}(t) = f(t, x(t), \mu, u(t)) \quad \text{where } x(t) \in \mathbb{R}^n \qquad \text{with output} \quad y(t) = g(t, x(t), \mu) \quad \text{where (e.g.) } y(t) \in \mathbb{R}, \tag{1}$$

and μ are system parameters. In the experiments of [2, 6, 7] the dynamical systems were forced oscillators, the output y was a position coordinate and the feedback control was in the form of a PD control, $u(t) = k_p[y(t) - y_\mathrm{r}(t)] + k_d[\dot{y}(t) - \dot{y}_\mathrm{r}(t)]$, with a reference time profile y_r. The control is said to be stabilizing the (for example) T-periodic trajectory $x_*(t)$ (with output $y_*(t) = g(t, x_*(t), \mu)$) of the uncontrolled system $\dot{x} = f(t, x, \mu, 0)$, if for every T-periodic reference $y_\mathrm{r} \approx y_*$ and initial conditions (t, x) close to $(t, x_*(t))$ the controlled system (1) converges to a unique T-periodic limit $y_\mathrm{lim}(t) \approx y_*(t)$. Moreover, the approach requires that the *asymptotic input-output map* $Y : (y_\mathrm{r}, \mu) \mapsto y_\mathrm{lim}$ is continuously differentiable in the space of T-periodic functions. If the stabilization condition is satisfied for the feedback control $y - y_\mathrm{r} \mapsto u$, then one may find the periodic orbit y_* of the uncontrolled system as fixed point of the map Y: $y_\mathrm{r} = Y(y_\mathrm{r}, \mu)$ if and only if $y_\mathrm{r} = y_*$, regardless of the dynamical stability of y_*. This enabled the authors of [1, 2, 5, 6, 7] to track response curves through limit (fold/saddle-node) bifurcations, track fold bifurcations in two parameters, and detect stable and unstable directions of saddle-type orbits in mechanical oscillator experiments.

Solving the nonlinear fixed-point problem $y = Y(y, \mu)$

One difficulty when solving for (or tracking) fixed points of the input-output map Y is that the Jacobian of $Y(y_\mathrm{r}, \mu)$ with respect to its arguments, which Newton iteration-based solvers require, is not known, and can generally be obtained only by performing repeated experiments for small deviations of the inputs, $(y_\mathrm{r} + \delta y_\mathrm{r}, \mu + \delta\mu)$. For mechanical oscillator experiments the periodic orbits are nearly harmonic such that [7] approximated y_r with low-order harmonics: $y_\mathrm{r}(t) \approx P_N[y_\mathrm{r}](t) := \sum_{\ell=-N}^{N} y_\ell b_\ell(t)$, where, in their case, $b_\ell(t) = \cos(\ell\omega t)$ for $\ell \leq 0$, $b_\ell(t) = \sin(\ell\omega t)$ for $\ell > 0$, $\omega = 2\pi/T$ and $N \leq 10$ typically. Barton, Renson *et al* used a (Newton-)Picard iteration, splitting $y_\mathrm{r} = y_P + y_Q$ with $y_p \in \operatorname{rg} P_1$ and $y_Q \in \operatorname{rg} Q_1$ ($Q_N = I - P_N$). They observed that, for fixed (y_P, μ), the iteration $y_Q \mapsto Q_1 Y(y_p + y_Q, \mu)$ converges to a limit y_Q within measurement accuracy in one or two iterations, defining a map $Y_Q(y_P, \mu)$. This reduced the fixed point problem to the low-dimensional $y_P = P_1 Y(y_p + Y_Q(y_P, \mu), \mu)$ in $\operatorname{rg} P_1$, for which a finite-difference approximation of the Jacobian is feasible.

We generalize this Newton-Picard approach to problems where we expect a severely non-harmonic fixed point y_*, that is, typically problems with large period T. Our illustrating example below considers a forced system with forcing as shown in fig. 1(top-left). The Picard iteration $y_Q \mapsto Q_N Y(y_P + Q_N y_Q, \mu)$ suffers a linear low-frequency instability for increasing periods T and fixed N. This is best illustrated considering the simplest case $\dot{x} = ax - k[y - y_\mathrm{r}]$ with $y = g(x) = x$ and $0 < a < k$ for (1) on an interval $[0, T]$. In this case the map Y is linear and commutes with P_N and Q_N, and the map $y \mapsto Yy$ has unstable eigenvalues corresponding to eigenfunctions of the form $\exp(2\pi \mathrm{i}\ell t/T)$ for all $\ell < T\sqrt{a(2k-a)}/(2\pi) =: m$. Thus, for the Picard iteration $y \mapsto Q_N Y(y_p + y)$ (with fixed y_P) to converge, the projection P_N must be injective on the space spanned by the m lowest harmonic modes. This criterion determines the necessary dimension of the space $\operatorname{rg} P_N$ of variables in which one has to formulate the nonlinear problem for the Newton iteration, which is in general high-dimensional for large periods T:

$$y_P = P_N Y(y_P + Y_Q(y_P, \mu), \mu) \quad \text{for } y_p \in \operatorname{rg} P_N, \text{ where } \dim \operatorname{rg} P_N \sim N \gg 1 \text{ for } T \gg 1, \text{ such that } N \sim T. \tag{2}$$

The problem can be addressed if the control law $y_\mathrm{r} - y \mapsto u$ stabilizes such that perturbations decay on a time horizon h of order 1 uniformly in $[0, T]$ (using the additional arguments in y to indicate initial time and initial condition for state x):

$$|y(t; t_0, x_1) - y(t; t_0, x_2)| \leq C \exp(-\gamma(t - t_0))|x_1 - x_2| \tag{3}$$

in (1) for $\gamma > 0$, C of order 1, independent of the period T. In this case perturbations at time t_0 do not have noticeable influence anymore at time $t_0 + h$ (where h is s.t. $C \exp(-\gamma h) \ll 1$). If criterion (3) is satisfied, we may choose for $\operatorname{rg} P_N$, for example, the space of piecewise constant functions: $[P_N y](t) = T/N \int_{t_{\ell-1}}^{t_\ell} y(s)\mathrm{d}s =: y_\ell$ if $t \in J_\ell = [t_{\ell-1}, t_\ell)$, where $t_\ell = \ell T/N$. The variable for the nonlinear problem (2) is then $(y_1, \ldots, y_N, \mu)$, and (2) poses an equation on each interval J_ℓ. Due to the finite-time decay condition (3), $[\partial/\partial y_\ell] P_N Y|_{J_\nu}$ is small if the distance between ν and ℓ satisfies $|\nu - \ell| > hN/T =: q = O(1)$. Hence the Jacobian $\partial P_N Y/\partial y_P$ has only q non-zero diagonals. This implies that deviations δy_ℓ and δy_ν can be applied simultaneously if $|\ell - \nu| > q$ when determining the finite difference approximation for $\partial P_N Y/\partial y_P$. Consequently, the fixed point problem (2) with a projection P_N chosen such that the Picard iteration is linearly stable on $\mathrm{rg}[I - P_N]$ can be solved with a (computational or experimental) effort that grows linearly in the period T because the number of necessary evaluations of Y is independent of the period T.

Example — quasiperiodically forced delay differential equation (DDE) modelling the Mid-Pleistocene transition
We demonstrate the feasibility for a simple quasiperiodically forced model for palaeoclimate ice ages, modelling the Mid-Pleistocene transition, which is a simplification of a model originally proposed by Saltzman & Maasch, see [3, 4],

$$\mathrm{d}x(t) = \left[-px(t-\tau) + rx(t) - sx(t-\tau)^2 - x(t-\tau)^2 x(t) - aI(t)\right]\mathrm{d}t + \sigma dW_t, \tag{4}$$

for the global ice mass anomaly x over the last 2 million years. Quinn *et al* [3, 4] observed that the forcing by variability of solar insolation $I(t)$, shown in fig. 1(top-left), causes a transition at time t_c from small-amplitude fluctuations around an equilibrium (at $x = -0.5$) to a large-amplitude limit cycle for forcing amplitudes a greater than some critical value a_c ($a_c = 0.1$ for transitions without noise). The time t_c is close to where the Mid-pleistocene transition from rapid to slow ice ages occurrs in data sets. Continuation of the saddle and the attractor for positive a without noise (using DDE-Biftool) shows that the two non-autonomous trajectories pinch at t_c. In the infinite-time limit, saddle and node form a strange non-chaotic attractor at the critical amplitude a_c. We track the saddle for the non-autonomous system (enforcing artificially periodic boundary conditions) as a test case for the control-based continuation of complex time profiles with random disturbances of size σ. Feedback control was trivially applicable by adding it to the solar insolation: $aI(t) + k[y_\mathrm{r} - y]$, where output $y = x$. A typical time profile is shown (in red) in fig. 1(bottom-left), the partial bifurcation diagram is in fig. 1(bottom-right). Note that saddle and node do not form a smooth saddle-node near $a = a_c$ without noise.

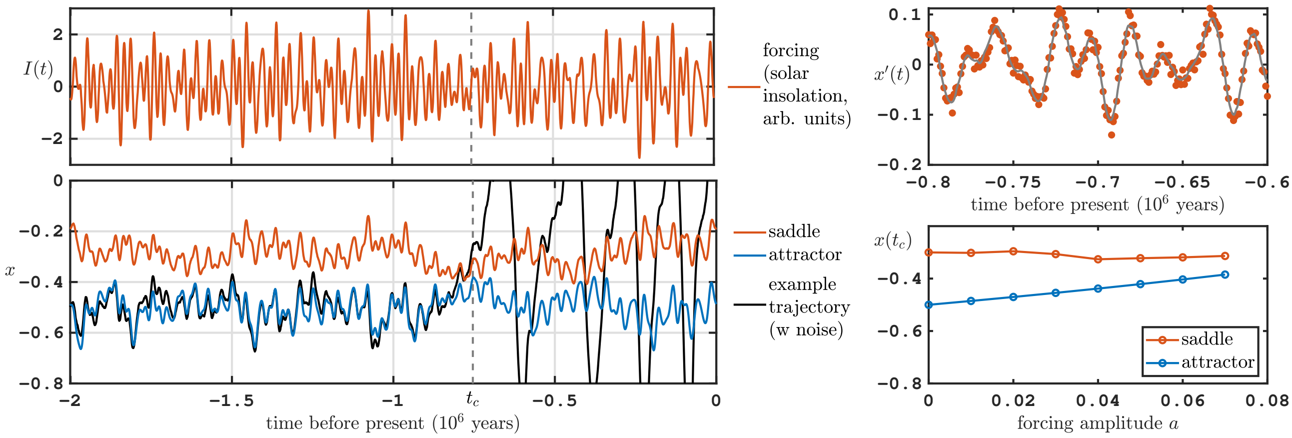

Figure 1: (top-left) Solar insolation $I(t)$ at 65° degree North in the summer [4]; (bottom-left) nonautonomous attractor and saddle, and large-noise ($\sigma = 6 \times 10^{-3}$) trajectory illustrating transition near time t_c, caused by saddle-node pinching; (top-right) approximate $\mathrm{d}x/\mathrm{d}t$ illustrating magnitude of disturbance; (bottom-right) partial bifurcation diagram for value of saddle and attractor at t_c. Parameters as in [3]: $p = 0.95$, $r = s = 0.8$, $\tau = 1.31$, Euler-Maruyama scheme stepsize 0.1, $\sigma = 3 \times 10^{-3}$, $N = 200$, gain $k = 2$.

Potential future experimental test cases are forced mechanical single-degree-of-freedom oscillators with hardening nonlinearity where one may track connecting orbits caused by brief spikes of the forcing amplitude.

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