Dynamics of Piecewise Linear Mathieu Equation with Non-Zero Offset

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Summary. The current work considers a piecewise linear Mathieu equation with the point of asymmetry being non-zero. The considered oscillator is essentially nonlinear and the primary objective of this study is to explore the regions of instabilities as a function of the excitation amplitude, frequency and interestingly the initial energy imposed on the oscillator. In this study we invoke energy-angle variables and the method of averaging to analytically describe the interesting energy dependent instability zones. We explore the instability zones in the vicinity of $1:1$ parametric resonance. We show that the derived analytical model provides a fairly good first order approximation to the unstable regions emerging in the vicinity of $1:1$ resonances and it is expected that the analytical method described is applicable to higher order resonances as well.

Introduction

Many computational and analytical attempts have been devoted to understanding of the complex dynamics of a 1DOF piecewise linear oscillator (PWLO) model subject to various types of excitations such as external and parametric forcing. Shaw and Holmes [1] have analysed semi-analytically the harmonically forced PWLO for the periodic orbits and their bifurcations. This study has been followed by Thompson et. al [2] who analysed numerically the subharmonic resonances, bifurcations and chaotic regimes of forced PWLO. Natsiavas [3] has considered the forced response of PWLO with bi-linear damping and successfully derived the relatively simple, semi-analytic solution for n-periodic orbits and analysed their stability. Using a similar approach, author has analysed the n-periodic response including stability analysis of PWLO incorporating the Van-der-Pol type damping [4]. Other works by Natsiavas et al. [5, 6] present the parametric excitation of piecewise linear oscillator as a typical model of gear backlash and asymmetric stiffness. A computational study by Chatterjee et al. [7] was performed on the model proposed by Natsiavas et al. [5] exploring the regions of instability. The current study dwells on the analytical description of these instability regions using asymptotic analysis. The resulting modulation equation provides a first order approximation of these instability boundaries in this class of systems and the approximations obtained by the considered method are found to be good in the case of sufficiently low excitation level.

Model and Analysis

Let us consider a piecewise linear (PWL) oscillator with a non-zero offset ($a > 0$) subject to a parametric excitation,

$$\ddot{q} + f(q) + \varepsilon P \cos(\Omega_e t) q = 0 \tag{1a}$$

$$f(q) = \begin{cases} q, q < a \\ \delta^2 q + (1-\delta^2)a, q \geq a \end{cases} \tag{1b}$$

The above equation is scaled such that the stiffness coefficient is unity for $q < a$, whereas the corresponding coefficient is $\delta \geq 1$ for $q \geq a$. In what follows δ will be referred to as the 'asymmetry parameter' and a as the 'offset parameter'. In fact, Eq. (1) or its different variants have been previously considered in several theoretical works for the two important limiting cases.

Unlike the linear Mathieu equation, its PWL counterpart with a non-zero offset, possesses the energy dependent parametric instability zones. To illustrate this fact, we plot the energy ($E(t)$) corresponding to Eq. (1) assuming the parametric excitation in the vicinity of $1:1$ resonance with the fixed level of excitation amplitude (P) and frequency for the two different initial energy levels ($E(0)$). As is clear from the results of Figure 1, formation of unbounded response strongly depends on initial energy. The main objective of the present study is to characterize analytically these energy dependent instability zones. This is done by reformulating the system in terms of the action (I) - angle (θ) (A-A) variables ($q(I,\theta)$), transforming to energy (E)-angle variables ($q(E,\theta)$) and averaging in the vicinity of ($1:n$) resonance. Subsequent analysis of the averaged equations reveals the formation of quite interesting, energy dependent instability regions for a general resonance condition ($1:n$). We end up with the following averaged flow,

$$\dot{E}_{Avg} = \frac{\varepsilon n P}{2\lambda}\{\alpha \sin(\upsilon) + \beta \cos(\upsilon)\} \tag{2a}$$

$$\dot{\upsilon} = \frac{n}{\lambda}\left\{1 + \frac{\varepsilon P}{2}\{\alpha' \cos(\upsilon) - \beta' \sin(\upsilon)\}\right\} - \Omega_e \tag{2b}$$

Where E_{Avg} – stands for the averaged energy, υ – resonance phase, $\lambda = dE/dI$, $\alpha(E), \beta(E)$ are the real and imaginary part of the Fourier coefficient $f_n(E)$ corresponding to $q^2(\theta, E)$ and α', β' are the corresponding derivatives with respect to energy. In Figure 3 (Left panel) we illustrate the phase plane of the averaged flow Eq. (2), while in Figure 3 (Right panel) we plot the transition curves on the force - energy plane corresponding to the saddle point of the averaged flow.

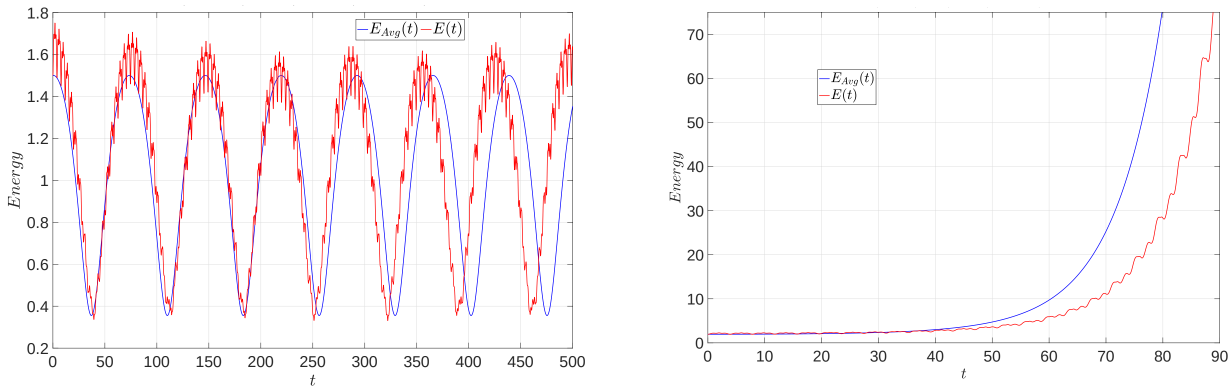

Figure 1: Energy as a function of time corresponding to $1:1$ resonance for $a = 0.3, P = 3, \delta = 4$ (Left panel) initial energy lower than the threshold ($E(0) < 1.858, \nu(0) = 3.71$) (Right panel) initial energy higher than the threshold ($E(0) > 1.858, \nu(0) = 3.71$). (refer to the phase contour in the right panel of Figure 2)

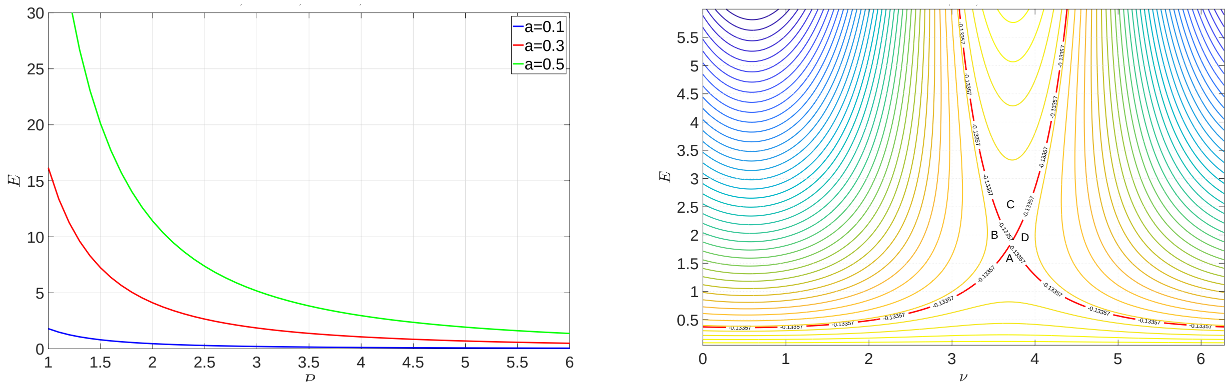

Figure 2: (Left panel) Transition force - energy curves (Right panel) Phase plane of the averaged flow corresponding to $P = 3, a = 0.3$ both correspond to $\delta = 4$

As is clear from the phase portrait of the averaged flow, the change in the initial phase will lead to a change in the critical value of initial energy above which the system will exhibit the unbounded response. This behavior is illustrated in Figure 2. One can observe a fixed point (saddle) corresponding to Eq. (2) and the phase plane is split into stable region (A) and unstable regions (B, C, D) delimited by the separatrix. If the initial conditions are picked in the region A, the oscillator will exhibit a bounded response. Whereas, the oscillator exhibits unbounded response for any initial excitation in the other three regions. The correspondence between the analytical approximation and the true model is found to be extremely good for the lower values of the excitation amplitude.

Conclusions

The current analytical and numerical study is devoted to the analysis of the response of piecewise linear Mathieu equation with non-zero offset. To obtain some relatively simple analytical description of the instability zones of a PWL Mathieu equation, we introduce the action-angle variables and apply the method of averaging to deduce a slow-flow model corresponding to a specific resonance case. The study of the slow-flow model provides a clear description of the instability zones. As we have already shown above, these boundaries are not only dependent on the excitation amplitude but also on the initial conditions. The numerical simulations of the full model match extremely well with the deduced slow-flow model for the lower values of the amplitude of parametric excitation. However, the presented analytical model is not devoid of its drawbacks in the sense that it fails to predict the instability boundaries for the higher values of the forcing amplitude. As we have already noted above, in that case the mechanism which leads to unbounded response for essentially low values of energy, is strongly chaotic and cannot be described by the constructed averaged flow.

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