# Escape Dynamics of a Parametrically Excited Particle from an Infinite Range Potential

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*Summary*. The current papers dwells on the classical problem of escape from a potential well. The considered potential well in this study is infinite range, with one global minimum and reaching maximum asymptotically at infinity. The particle is parametrically excited and we consider the threshold of escape of this particle from the bottom of the well. The analytical study is based on invoking the canonical action-angle variables and the averaged dynamics on a resonance manifold.

## Introduction

The oscillatory behavior of a dynamical system is governed by the potential to which the system is subjected to. In order to study the dynamics of the system close to the center (local minimum in a conservative system), one could linearize the system and add the nonlinear terms as weak perturbations. However, such quasilinear dynamical systems seldom model the behavior away from the center. For example, the classical case of escape dynamics of a forced particle from a potential well can hardly be analytically studied using a weakly nonlinear model, since problem of escape is essentially a transient phenomenon. Such problem are plenty in the domain of applied physics and engineering and have been quite extensively studied [1]. From an engineering perspective, the dynamical behavior of capsizing of sea vessels is provided by Virgin [2], whereas the dynamic pull-in and the escape dynamics thereof in MEMS devices were reported by Younis et al. [3], studies by Virgin et al. [4] considers escape from a well under harmonic excitation, while Mann [5] considers an energy-based criterion for escape from the double well of a magnetic pendulum. Recent studies by Gendelman et al. [6, 7] invoke the canonical action-angle (AA) variables and averaging [8] to study the escape. Current study is based on the framework of AA variables.

As such, an analytical prediction of the particle escape from the potential well becomes quite relevant. We consider an undamped particle oscillating in an infinite-range potential. The minimum of the well is Lyapunov stable and the fixed point persists even with the application of parametric excitation. However, the excitation can render the fixed point unstable, but may not necessarily lead to escape. In contrast, for certain parameter range (excitation frequency and amplitude) one can observe escape (ref. Fig. 1) of the particle. The objective herein is to numerically and analytically study the parameter range of amplitude and frequency that leads to escape and the route thereof. The escape is characterized by breaching of the separatrix (connecting the two fixed points at $(\pm\infty, 0)$) as shown in Fig. 1.

**Mathematical modeling and analysis**

Consider the dynamics of a parametrically excited (amplitude $\varepsilon$ and frequency $\Omega$) particle in an infinite-range potential

$$\ddot{q} + \frac{\partial V(q,t)}{\partial q} = 0;\ V(q,t) = -\frac{\{1+\varepsilon \sin(\Omega t)\}}{2\cosh^2(q)} \tag{1}$$

We are interested in the transition values of $(\Omega, \varepsilon)$ that would render a particle situated close to the bottom (but not exactly at the bottom) of the well to escape. To this end, we introduce the AA $(I, \theta)$ variables and the canonical transformation resulting in $p = p(I, \theta), q = q(I, \theta)$. The perturbed Hamiltonian in AA variables is

$$\mathcal{H}(I,\theta,t) = \mathcal{H}_0(I) - \varepsilon \sin(\Omega t)\{1 + C^2 \sin^2(\theta)\}^{-1}/2 \tag{2}$$

Where $\mathcal{H}_0(I) = -(1-I)^2/2, C = \sqrt{2I - I^2}/(1-I)$ and $I = 0$ corresponds to the bottom of the well. Since $q(I,\theta)$ is $2\pi$ periodic, the perturbation term in Eq. 2 can be expanded in Fourier series. We consider $2:1$ resonance and thereby introduce slow phase variable $\vartheta = 2\theta - \Omega t$. On averaging the slow-flow equation corresponding to the evolution of AA variables over the fast phase variables, we have the averaged $(I \mapsto J, \vartheta \mapsto \psi)$ slow-flow equations in the form,

$$\begin{gathered}\dot{J} = -\varepsilon J\{1 + (J-2)^{-1}\}\cos(\psi) \\ \dot{\psi} = 2(1-J) + \varepsilon\{1 - 2(J-2)^{-2}\}\sin(\psi) - \Omega\end{gathered} \tag{3}$$

The fixed points of the slow-flow equations are readily calculable to be (i) $J = 0, \sin(\psi) = 2(\Omega - 2)/\varepsilon$, (ii) $J = 1, \sin(\psi) = -(\Omega/\varepsilon)$ (iii) $\psi = \pi/2, 3\pi/2, \{2(1-J) - \Omega\}(J^2 - 4J + 4) \pm \varepsilon(J^2 - 4J + 2) = 0$ respectively. The fixed point at $\psi = \pi/2, J > 0$ is a center and $\psi = 3\pi/2, J > 0$ is a saddle for $\Omega < 2$. For a specific value of $\Omega < 2$, with an increase in the excitation amplitude, the saddle goes through a pitchfork bifurcation and there is emergence of two additional fixed points (i). Upon bifurcation, fixed point at $\psi = 3\pi/2, J < 0$ is a center, but not of much significance. The bifurcations would become apparent by investigating the integral of motion of the averaged system given by

$$\mathcal{M}(J,\psi) = -(1-J)^2 + \varepsilon J(J-1)(J-2)^{-1}\sin(\psi) - J\Omega \tag{4}$$

The bifurcation point corresponds to the threshold of the excitation amplitude resulting in escape. The locus of these points is shown in Fig. 2 as red curves ($LB_{avg}$, $RB_{avg}$) emanating from $\Omega = 2$. Incidentally, these curves also correspond to the instability boundary of the Mathieu equation $\ddot{x} + x\{1 + \varepsilon \sin(\Omega t)\} = 0$, resulting from the linearization of Eq. 1 about (0,0). The escape threshold for the exact system (Eq. 1) is indicated as $LB_{ex}$, $RB_{ex}$. As observable, there is the close match of the $LB_{avg}$ and $LB_{ex}$. However, the right boundary shows a very distinct transition wherein the averaged system predicts a much lower escape threshold in comparison to the actual dynamical system (Eq.

1). It is noted that along the boundary predicted by the averaged system, the center corresponding to the bottom of the potential well bifurcates to a saddle. This is evidenced by the Poincare maps of Fig. 3 ($\Omega = 1.8$). In fact, along $LB_{ex}$, the creation of this saddle leads to the particle escape. In contrast, in the regions other than $LB_{ex}$, the creation of the saddle (along $LB_{avg}, RB_{avg}$) leads to chaotic motion albeit bounded. On further increase in the excitation amplitude, the tori are broken and the particle escapes. This behavior is observable in the Poincare maps of Fig. 4 ($\Omega = 1.9$). The minimum force amplitude required for the escape corresponds to a frequency $\Omega < 2$ ($\Omega \cong 1.85$) and is owing to the fact that Eq. 1 exhibits softening nonlinearity.

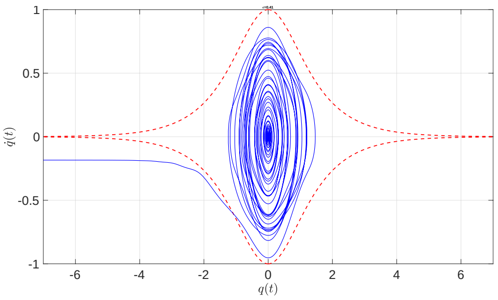

Figure 1: Particle escape from the bottom ($q(0) = 10^{-6}, \dot{q}(0) =$ *0* of the potential well for $\varepsilon$*=0.41,*$\Omega$*=1.8*. (Broken red curves correspond to the separatrix)

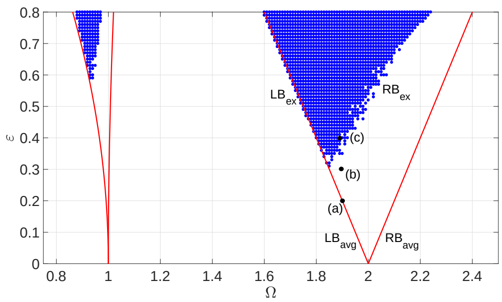


Figure 2: Escape threshold. Blue shaded region (num. simulation of Eq. 1) corresponds to particle escape (for $t \leq 6000$). The red lines emanating from $\Omega = 2$ correspond to the point of bifurcation corresponding to Eq. 3 and those emanating from $\Omega = 1$ correspond to the Mathieu equation described above

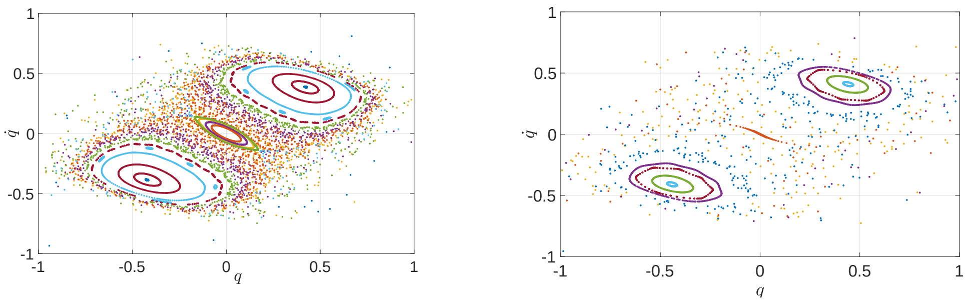

Figure 3: Poincare maps corresponding to $\Omega = 1.8$ (left panel) $\varepsilon = 0.3$ (right panel) $\varepsilon = 0.39$

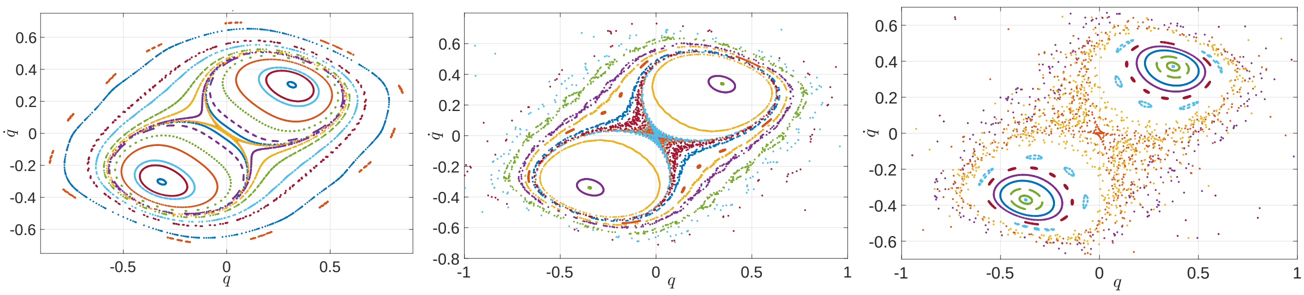

Figure 4: Poincare maps corresponding to $\Omega = 1.9$ (a) $\varepsilon = 0.2$, (b) $\varepsilon = 0.3$, (c) $\varepsilon = 0.4$ (a-c refers to the points indicated in Fig. 2)

## Conclusions

The current study considers the escape dynamics of a parametrically excited particle (located close to the bottom of the well) from an infinite-range potential. We invoke the AA variables and study the dynamics of the system on the resonance manifold and predict the threshold for escape as a function of frequency. For certain $(\Omega, \varepsilon)$, the bifurcation of the bottom of the well to a saddle leads to the escape, whereas for other parametric range, the escape is through a chaotic route as evidenced by the Poincare maps and the slow-flow model fails to predict the escape in this case.

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