Nonsmooth Modal Analysis of Varying Cross-section Bar

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Summary. The nonsmooth modal analysis of a simple one-dimensional bar of constant cross-section has already been successfully performed using a Finite Volume formulation and the Frequency-Domain Boundary Element Method (FD-BEM). Both strategies took advantage of the existence of d'Alembert solution for such problem. The present contribution extends the previous works to a bar of non-constant cross-section for which the d'Alembert solution no longer exists. The proposed scheme combines the finite element method in space to the harmonic balance technique in time. The solution satisfies the unilateral contact condition along with an energy-preserving implicit impact law in a weighted-residual sense. The partial backbone curve of the first mode shows the existence of internal resonances.

Introduction

Within the framework of structural dynamics, modal analysis is a practical tool to predict the occurrence of vibrational resonances, most commonly to prevent them. Various formulations have been proposed in the nonlinear framework where the governing equation contains smooth nonlinear function of the state of system. Nonsmooth modal analysis is one incarnation of nonlinear modal analysis in which the smoothness assumption does not hold, as for instance exhibited in unilaterally-constrained structural dynamics [1, 2]. In this context, a method combining the finite element method (FEM) and harmonic balance method (HBM) is proposed.

System of Interest

The system of interest, in Figure 1, is a one-dimensional bar of finite length L. The displacement field of straight cross-sections is denoted $u(x,t)$ where x is the space coordinate and t is time. The bar is clamped at $x = 0$ (Dirichlet condition) and subject to a unilateral contact condition at $x = L$ (Signorini condition). The cross-section area $A(x)$ is varying along the bar while the mass density ρ and Young's modulus E are constant. Given the initial gap g_0 between the bar at rest and the rigid foundation, the gap function reads $g(t) = g_0 - u(L,t)$. The dynamics of the bar is governed by the Partial Differential Equation

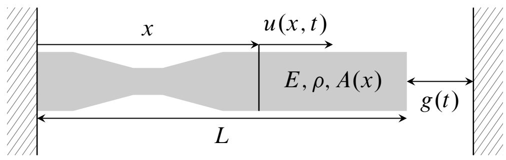

Figure 1: One-dimensional bar system of interest

$$EA_x u_x + EAu_{xx} + \rho A u_{tt} = 0 \tag{1}$$

where $\bullet_\eta$ denotes a partial differentiation with respect to η. Clamping at $x = 0$ reads $u(0,t) = 0$ while unilateral contact at $x = L$ is expressed as

$$-\lambda(t) + \max[\lambda(t) - \alpha g(t), 0] = 0 \tag{2}$$

where $\lambda(t)$ is the contact force and α is a strictly positive real number.

Solution method: Fourier Transform+Finite Elements+Harmonic Balance

The sought families of periodic solutions defining the modal motions are computed through a numerical scheme which assumes that no impact law is explicitly required, in contrast to common practises. First, a Fourier Transform is applied on the unknowns of the problem:

$$\mathcal{F}\{u(x,t)\} = \hat{u}(x,\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,t)\exp(-i\omega t)\mathrm{d}t \quad \text{and} \quad \mathcal{F}\{\lambda(t)\} = \hat{\lambda}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lambda(t)\exp(-i\omega t)\mathrm{d}t. \tag{3}$$

This leads to the two governing equations in the frequency domain

$$EA\hat{u}_{xx} + EA_x\hat{u}_x - \omega^2\rho A\hat{u} = 0 \quad \text{and} \quad -\hat{\lambda}(\omega) + \mathcal{F}\{\max[\lambda(t) - \alpha g(t), 0]\} = 0 \tag{4}$$

where the last term $\mathcal{F}\{\ldots\}$ can unfortunately not be explicitly expressed in terms of $\hat{u}$ and $\hat{\lambda}$. Spatial semi-discretization is applied via the standard FEM with N two-node linear elements and the corresponding $N+1$ nodes. The Dirichlet boundary condition is directly enforced in the discretized weak form of (4) which reads

$$\mathbf{H}(\omega)\hat{\mathbf{u}}(\omega) = \hat{\mathbf{f}}(\omega) \tag{5}$$

with $\mathbf{H}(\omega) = \mathbf{K} - \omega^2\mathbf{M}$ where $\mathbf{K}$ and $\mathbf{M}$ are the classical stiffness and mass matrices. Also, vector $\hat{\mathbf{f}}(\omega) = [0,\ldots,0,\hat{\lambda}(\omega)]^\top$ stores the contact force while vector $\hat{\mathbf{u}}(\omega) = [\hat{u}_0(\omega),\ldots,\hat{u}_{N-1}(\omega),\hat{u}_N(\omega)]^\top$ stores the response nodal displacements, both in the frequency domain. Inverting $\mathbf{H}(\omega)$ yields the relation

$$\hat{u}_N(\omega) = \gamma(\omega)\hat{\lambda}(\omega) \tag{6}$$

where $\gamma(\omega)$ is the last entry of the inverse of $\mathbf{H}(\omega)$. Since periodic solutions are targeted, the time-domain nodal displacements and companion contact force are approximated as

$$\lambda(t) = a_0 + \sum_{k=1}^{M} a_{2k-1} \sin k\Omega t + a_{2k} \cos k\Omega t, \quad u_i(t) = b_{i0} + \sum_{k=1}^{M} b_{i2k-1} \sin k\Omega t + b_{i2k} \cos k\Omega t, \quad i = 1, \ldots, N. \quad (7)$$

Given the form of $\hat{\mathbf{f}}$, the only unknowns of the problem actually are the Fourier coefficients of $u_N(t)$ and $\lambda(t)$, that is the coefficients (a_k, b_{Nk}) for $k = 0, \ldots, 2M$. Accordingly, it is required to establish the corresponding equations to solve for. First, inserting (7) into (6) leads to

$$b_{Nk} = \gamma(k\Omega) a_k, \quad k = 0, \ldots, 2M. \quad (8)$$

Second, noting $\beta(t) \overset{\text{def}}{=} \max(\lambda(t) - \alpha g(t), 0)$ and $T = 2\pi/\Omega$, the HBM version of condition (2) implies

$$a_0 - \frac{1}{T} \int_0^T \beta(t) \mathrm{d}t = 0 \quad (9)$$

along with

$$a_{2k} - \frac{2}{T} \int_0^T \beta(t) \cos k\Omega t \mathrm{d}t = 0 \quad \text{and} \quad a_{2k-1} - \frac{2}{T} \int_0^T \beta(t) \sin k\Omega t \mathrm{d}t = 0, \quad k = 1, \ldots, M \quad (10)$$

where Expressions (7) are first inserted into the above integrals which can then be numerically evaluated using basic quadrature schemes. Continuous families of periodic solutions are built via a classical sequential continuation technique [3, 4] on the frequency parameter Ω, which is thus not treated as an unknown. The resulting system of $4M + 2$ equations (8) to (10) in $4M + 2$ unknowns can be solved using a nonlinear solver. Since autonomous periodic solutions are targeted, it seems justified to say that the classically required energy-preserving impact law is here embedded implicitly in the above integrals. However, we do not have a formal proof of this statement.

Results

The selected cross-section area is $A(x) = 1.5 - x$ with $L = 1$ and the initial gap is $g_0 = 0.001$. The discretization is chosen to be $M = 40$ and $N = 4000$. Results on the first nonsmooth mode are depicted in Figure 2. The backbone

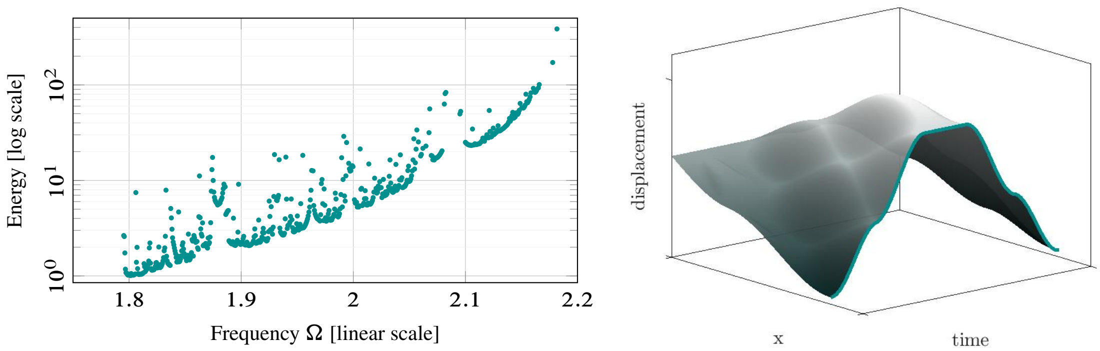

Figure 2: First nonsmooth mode: backbone curve (left) and displacement field at $\Omega = 1.907$ (right).

curve shows the frequency-energy dependency of the modal response and internal resonance mechanisms with higher modes. Unlike the displacement field of the constant cross-section bar which consists of piecewise affine segments, the displacement field found here seems to have smoother functional properties, even with a fine approximation.

Conclusion

A numerical scheme combining FEM and the HBM is proved capable of capturing nonsmooth modes for a one-dimensional varying cross-section bar. The main feature of the proposed scheme is that neither explicit energy-preserving impact laws nor regularization techniques are required at the contact interface even-though classical finite-elements are employed. Instead, the Signorini condition and companion impact law are satisfied in a weighted-residual sense.

References

[1] C. Yoong, A. Thorin and M. Legrand, "Nonsmooth modal analysis of an elastic bar subject to a unilateral contact constraint", *Nonlinear Dynamics*, 2018.

[2] T. Lu and M. Legrand, "Nonsmooth Modal Analysis via the Boundary Element Method for one-dimensional bar systems", *Nonlinear Dynamics*, 2021.

[3] M. Peeters, R. Viguié, G. Serandour, G. Kerschen, and J.-C. Golinval "Nonlinear normal modes, Part II: Toward a practical computation using numerical continuation techniques", *Mechanical Systems and Signal Processing*, 2009.

[4] A. H. Nayfeh and Balachandran, B. *Applied nonlinear dynamics: analytical, computational, and experimental methods*. Wiley, 2008.