Synchronization of networks of dynamical systems by nonlinear integral couplings and sequential decoloring of the network graph

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Summary. In this work we present a method for controlled synchronization of networked nonlinear systems based on nonlinear integral couplings. For a class of nonlinear systems and network topologies, this method allows one to design synchronizing nonlinear couplings with noticeably lower coupling gains (understood in a nonlinear sense) than for the case of linear diffusive couplings. This results in lower control values and energy consumption needed for synchronization as well as lower sensitivity to measurement noise. The method is illustrated by application to synchronization of Hindmarsh-Rose oscillators.

Synchronization and nonlinear integral couplings

In this abstract we consider N identical nonlinear systems of the form

$$\dot{x}_i = f(x_i) + Bu_i, \;\; y_i = Cx_i, \;\; i = 1, \dots N, \tag{1}$$

with $x_i \in \mathbb{R}^n$, $y_i, u_i \in \mathbb{R}$, C^1 function $f(x)$ and matrices B and C of appropriate dimensions. The problem of controlled synchronization considered in this abstract is to find control laws for each u_i such that for any initial conditions of the closed-loop system, state vectors $x_i(t)$, $i = 1, \dots, N$ are bounded and

$$|x_i(t) - x_j(t)| \to 0, \;\; \text{as} \;\; t \to \infty, \;\; \forall i, j. \tag{2}$$

For each system i, u_i is allowed to depend on the system's output y_i and on the outputs y_j of the systems $j \in \mathcal{N}_i$, where $\mathcal{N}_i$ is the set of systems that can communicate to system i. The sets $\mathcal{N}_i$ specify the communication graph $\mathcal{G}$ for the network (there is an edge from node j to node i if $j \in \mathcal{N}_i$). It is required that for identical outputs $y_1 = y_2 = \dots = y_N$, the controls satisfy $u_1 = u_2 = \dots = u_N = 0$, such that in exact synchrony the systems exhibit dynamics of the unforced (with zero input) system (1).
In this work we propose synchronizing control laws u_i in the form of nonlinear integral couplings:

$$u_i = \sum_{j \in \mathcal{N}_i} \int_{y_i}^{y_j} \lambda(s) ds, \;\; i = 1, \dots, N, \tag{3}$$

where $\lambda(s) \geq 0$, $\forall s$, is a nonlinear coupling gain—the main design parameter in this scheme. Note that (3) is a generalization of conventional linear diffusive coupling, which is obtained from (3) with constant $\lambda(s) \equiv C$.
Contrary to the linear diffusive coupling, nonlinear integral coupling (3) allows one to differentiate the coupling strength depending on the location of the system's outputs in space, applying higher coupling gains only where the systems' nonlinearities counteract synchronization, while employing lower (or even zero) coupling gains where the nonlinearities do not have significant negative effects on synchronization. From the closed-loop performance point of view, the immediate consequences of this flexibility can be lower average synchronizing gains, lower coupling actions and energy needed to achieve and maintain synchronization and, consequently, lower sensitivity to measurement noise in outputs y_i. From the analysis point of view, this can provide better estimates of what should be the minimal coupling needed for synchronization. As couplings in physical, engineering and biological networked systems can have nonlinear nature (with their linear approximation studied in diffusive coupling), such estimates can shed more light on how synchronization is or can be achieved in such systems.

Main result

Let us first formulate notions and assumptions that delineate the considered class of systems (1) and the class of network topologies. The first assumption specifies the class of systems (1).

Assumption 1 *There exist* $P = P^T > 0$*,* $R = R^T > 0$ *and a scalar continuous function* $\gamma(s)$ *such that*

$$P \frac{\partial f}{\partial x}(x) + \frac{\partial f^T}{\partial x}(x) P - 2C^T C \gamma(Cx) \leq -R, \quad PB = C^T, \quad \forall x \in \mathbb{R}^n, \tag{4}$$

These conditions are satisfied for a class of incrementally minimumphase nonlinear systems [1]. Next, we formulate conditions on the communication graph $\mathcal{G}$. The conditions are linked to the notion of relaxed balanced coloring of the nodes of $\mathcal{G}$.

Definition 1 *A coloring of the nodes with $k \in \{1, \ldots, N\}$ colors $c_1, \ldots, c_k$ is called a* relaxed balanced coloring *if each node is assigned a color, and every c_i-colored node receives an equal number of edges from c_j-colored nodes for all $j \in \{1, 2 \ldots, k\} \backslash \{i\}$.*

A communication graph can be colored according to relaxed balanced coloring in multiple ways. The two trivial colorings are given by a) assigning each node an individual color and b) by assigning all nodes the same color. We call the graph *sequentially decolorable* if there exists a sequence of $N-1$ relaxed balanced colorings, starting from the coloring a) and ending with coloring b), such that each coloring is obtained from the previous one by taking two groups of nodes with two different colors are assigning them the same color. Examples of sequentially decolorable graphs include graphs with two nodes with bi- and unidirectional couplings; bi-directionally coupled rings with 3 and 4 nodes, a graph with N nodes with all-to-all couplings. There are algorithms that allow one to verify whether a graph is sequentially decolorable in a computationally efficient way. Now we can formulate our main result.

Theorem 1 *Consider N systems of the form* (1) *satisfying Assumption 1 and interconnected through nonlinear integral coupling* (3) *with a sequentially decorolable communication graph $\mathcal{G}$. Suppose the coupling gain function $\lambda(s)$ satisfies*

$$\lambda(s) \geq max(0, \gamma(s)), \ \ \forall s \in \mathbb{R}, \quad \int_{-\infty}^{+\infty} \lambda(s) ds \leq +\infty. \tag{5}$$

Then all solutions of the closed-loop system (1), (3) *are bounded and satisfy* (2).

Remark: The condition (5) can be relaxed by taking into account quantitative characteristics of communication graph. For example, for a graph with N nodes and all-to-all interconnections, the first condition in (5) can be substituted by $\lambda(s) \geq max(0, \gamma(s)/N)$.

Synchroniztion of Hindmarsh-Rose oscillators

We demonstrate our results with synchronization of Hindmarsh-Rose oscillators, which represent a simplified model of neuron dynamics [2]:

$$\dot{z}_1 = c - dy^2 - z_1, \ \ \dot{z}_2 = \varepsilon(m(y + y_0) - z_2), \quad \dot{y} = -ay^3 + by^2 + z_1 - z_2 + I + u, \tag{6}$$

where y, z_1 and z_2 represent various states of a neuron and external stimulation is provided by input u. All other parameters are positive constants. Analysis of synchronization in a network of such oscillators with *linear* coupling is presented in [3]. For numerical simulations we choose the following values of system parameters: $a = 1$, $b = 3$, $c = 1$, $d = 5$, $m = 4$, $I = 2.8$, $y_0 = 1.618$, $\varepsilon = 0.005$ [3]. In this abstract, we consider 4 systems interconnected in the following way: $\mathcal{N}_1 = \{2, 4\}$, $\mathcal{N}_2 = \{3\}$, $\mathcal{N}_3 = \{2, 4\}$, $\mathcal{N}_4 = \{1\}$. System (6) satisfies Assumption 1 with some $P = P^T > 0$, $R = R^T > 0$ and $\gamma(s) = \epsilon - 3as^2 + 2bs + \frac{(1-\delta ds)^2}{2(\delta - \epsilon)}$, for any sufficiently small $\epsilon > 0$ and, for the chosen system parameters, with $\delta = 0.2$, [1]. Thus, if we select $\lambda(s) = max(0, \gamma(s))$, the function $\lambda(s)$ will satisfy (5) and, by Theorem 1 synchronization will be achieved. Simulations results are shown in Figure 1. The results demonstrate the synchronizing system states y_i, z_{1i}, z_{2i} and control inputs u_i, $i = 1, \ldots 4$. The last plot shows variable gain $g_{12}(t)$ of the nonlinear integral coupling defined as $g_{12}(t) = \int_{y_1(t)}^{y_2(t)} \lambda(s) ds / (y_2(t) - y_1(t))$, [1]. According to the simulations, the gain varies from 3 down to 0. The higher coupling gain is applied whenever it is needed to achieve synchronization. It is reduced and even set to zero in accordance with system's dynamics while maintaining synchronization. The average gain over the simulation of 500s equals 1.09 (shown as red dashed line in Figure 1). The lower coupling gains (instantaneous and average) is a distinctive feature of the proposed method over linear diffusive couplings. The best estimate of the linear diffusive coupling gain that we are aware of is 3, which can be computed using the results of [4].

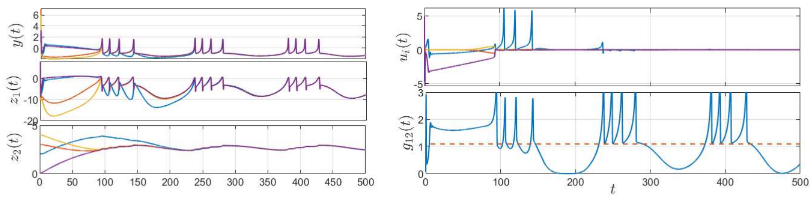

Figure 1: Synchronization of 4 Hindmarsh-Rose oscillators.

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