Time integration based stability calculation for delayed periodic system with linear time complexity

Daniel Bachrathy*,Kristof Nagy*
* *MTA-BME Lendület Machine Tool Research Group, Department of Applied Mechanics, Budapest University of Technology and Economics*

Summary. In this work, the implicit subspace iteration method is combinded with direct time integration of linear periodic delay differential equations. The linear time complexity of the proposed method is shown for different DDE solvers with higher order convergence. The propoesed method is really efficinet for dynamical system with large time period.

Semi-Discretization Method

There have been many algorithms [1, 5, 6, 7] developed during the last 20 years for systems where the time-delay effect is coupled with parametric excitation. The semi-discretization method (SDM) is one of the efficient ways to the analyze stability [1, 2]. The basic idea of the SDM is the numerical discretization of the delayed terms only above the delay-time interval. Consequently, the governing DDE becomes an ordinary differential equation (ODE) that can be solved in closed form in linear cases for each time step within these discrete delay-intervals, and a linear discrete map is constructed that describes the connection of the discretized state as a large vector over the discrete time instants within the delay interval. The size of this mapping depends on the resolution of the delay discretization.
If explicit time-periodicity also appears due to the parametric excitation, the time-periodic coefficients (or even the delays) should also be discretized in time over the time period, and the above procedure leads to different linear mappings at each sampled instant of the time period:

$$\mathbf{z}_{i+1} = \mathbf{G}_i \mathbf{z}_i \quad i = 0, 1, \dots n-1 \tag{1}$$

Here, $\mathbf{G}_i$ denotes the coefficient matrix connecting states $\mathbf{z}_i$ and $\mathbf{z}_{i+1}$, which are the vectors of the discretized states sampled at the discrete time intervals at subsequent time instants. A discrete map can be defined between the initial delay-discrete state $\mathbf{z}_0$ and the one $\mathbf{z}_m$ a principal period later:

$$\mathbf{z}_n = \mathbf{G}_{m-1} \dots \mathbf{G}_2 \mathbf{G}_1 \mathbf{G}_0 \mathbf{z}_0 = \mathbf{\Phi} \mathbf{z}_0, \tag{2}$$

where the transition matrix $\mathbf{\Phi}$ is a finite-dimensional approximation of the infinite-dimensional monodromy operator. Thus, the stability analysis is reduced to the problem whether the absolute values of all the eigenvalues of $\mathbf{\Phi}$ are less than one: $|\mu_i| < 1$. In order to improve numerical accuracy, the delay resolution, the order of semi-discretization, and the time periodicity resolution can be increased.

Implicit Subspace Iteration

Consider the general eigenvalue problem [4] $\mathbf{\Phi S} = \mathbf{S}\boldsymbol{\mu}$, where $\mathbf{\Phi}$ is an $\mathbb{R}^{n\times n}$ square matrix, $\boldsymbol{\mu}$ is a diagonal matrix of size $\mathbb{R}^{n\times n}$ containing the eigenvalues of $\mathbf{\Phi}$ on its main diagonal, and matrix $\mathbf{S}$ of size $\mathbb{R}^{n\times n}$ consists of the eigenvectors of $\mathbf{\Phi}$ in its columns. A set of $N_s < n$ dominant eigenvectors corresponding to the first N_s eigenvalues of the largest absolute values can be approximated in an iterative way. Let $\mathbf{S}_j$ of size $\mathbb{R}^{n\times N_s}$ denote the matrix of the N_s dominant eigenvectors after the jth iteration step. Starting from a random set of initial conditions $\mathbf{S}_0$ and taking an iteration, a new set $\mathbf{V}_j$ of size $\mathbb{R}^{n\times N_s}$ can be calculated according to the following iteration [4]

$$\mathbf{V}_j = \mathbf{\Phi} \mathbf{S}_j \tag{3}$$

$$\mathbf{V}_j \approx \mathbf{S}_j \mathbf{H}_j \;\rightarrow \mathbf{H}_j = \left(\mathbf{S}_j^{\mathrm{H}} \mathbf{S}_j\right)^{-1} \mathbf{S}_j^{\mathrm{H}} \mathbf{V}_j \tag{4}$$

$$\mathbf{H}_j = \mathbf{G}_j \boldsymbol{\lambda}_j \mathbf{G}_j^{-1} \tag{5}$$

$$\mathbf{S}_{j+1} = \mathbf{V}_j \mathbf{G}_j \tag{6}$$

where $\mathbf{G}_j$ is the matrix formed by the eigenvectors of $\mathbf{H}_j$ and $\boldsymbol{\lambda}_j$ is a diagonal matrix of eigenvalues of $\mathbf{H}_j$. Note, that in a numerical implementation, the approximated eigenvectors should be normalized after each iteration steps.
If $\mathbf{S}_j$ is obtained after a sufficient number of iteration steps, it will converge to the dominant eigenvectors, and the basis formed by the column vectors in both $\mathbf{S}_j$ and $\mathbf{V}_j$ span approximately the same space. Therefore, an approximate matrix $\mathbf{H}_j$ of size $\mathbb{R}^{N_s\times N_s}$ connecting $\mathbf{S}_j$ and $\mathbf{V}_j$ can be obtained using a pseudo-inverse calculation from the relation. After several iteration steps, the eigenvalues of $\mathbf{H}_j$ provide a good approximation for the dominant eigenvalues of $\mathbf{\Phi}$. This way, it is enough to compute the eigenvalues of a significantly reduced N_s-sized matrix $\mathbf{H}_j$ instead of the large n-sized matrix $\mathbf{\Phi}$. The details of this iteration process can be found in [3, 4].
In this form, the Implicit Subspace Iteration iteration is just an iterative method to find the eigenvalues of a known matrix $\mathbf{\Phi}$, however, the calculation of this matrix itself is the most time consuming operation if the time period is long $m \gg n$.

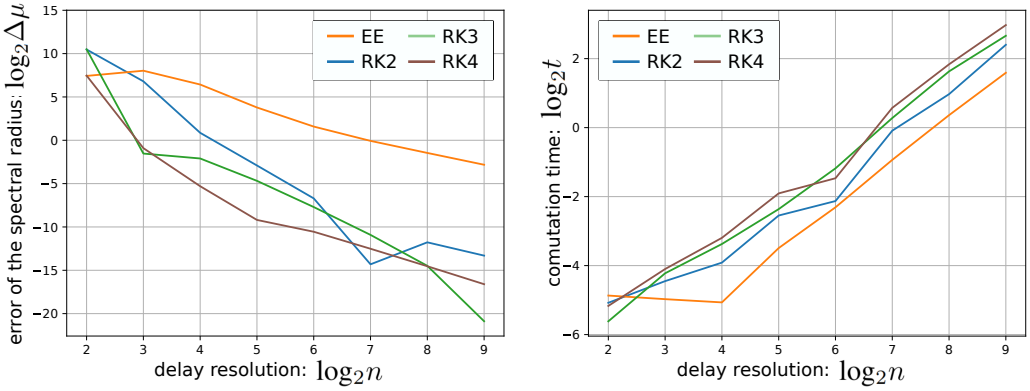

Figure 1: Convergence of thee spectral radius for the tradition SDM and for Implicit Sub Space Iteration combined with time integration with Explicit Euler (EE) method and higher-order Runge-Kutta methods

Dute to the m number of matrix multiplication in Eq.(2), the time complexity is proportional to $\mathcal{O}(mn^3)$, if a spares representation of $\mathbf{G}_i$ is used then it can tend to $\mathcal{O}(mn^2)$ [8, 9]. To decrease the computational need one can reduce the stepsize (m and n together) and increase the order of semi-discretization, however, it is complicated to increase it above 2 [2].

Direct time integration

It is also possible to determine the mapping in Eq.(3) without calculating the whole matrix $\mathbf{\Phi}$. The vector $\mathbf{V}_j$ can be calculated directly by means of the time integration of the equations of motion of the given system with $\mathcal{O}(m)$ time complexity. Thus, the advantage of implementing the ISIM is that the transition matrix $\mathbf{\Phi}$ does not have to be calculated. During the time integration process, one can use the same approximation as in the SDM (constant coefficients within the time step), then the final results for the eigenvalues will be identical.
However, if we apply time integration, then any advanced higher-order fixed-time-step computation scheme can be used to increasre the reat of convergence of the eigenvalues, while the time complexity will be the same. This is presented for the well-known delayed Mathieu equation:

$$\ddot{x}(t) + \kappa\dot{x}(t) + (\delta + \epsilon cos(\omega t))\, x = bx(t-\tau). \tag{7}$$

In Fig.1 the error of the largest multiplier is shown for paramteres $\kappa = 0.01, \delta = 5, \epsilon = 1, b = 1, \tau = 2\pi, \omega = 0.01$ (note, that $m = 100n$). The reference values for the eigenvalues is computed of $n = 2^15$.

Conclusions

The combination of the IISI with fixed-step-time-integration have only a linear time complexity and the convergence rate of the eigenvalues is the same as the order of the integration scheme. The next research goal is to the implement integrators with variable step size, which will have a great advantage in the case when the change of the coefficients of the governing equation are not smooth, however, the non-uniform steps size leads to a difficulty in the computation of $\mathbf{H}_j$ in Eq.(4).

Acknowledgement

This paper was supported by the Hungarian Scientific Research Fund OTKA FK-124462 and PD-124646.

References

[1] T. Insperger, Full-discretization and semi-discretization for milling stability prediction: Some comments, *Int. J. Mach. Tool Manu.* (2010) **50**:658-662

[2] T. Insperger, G. Stepan, J. Turi, On the higher-order semi-discretizations for periodic delayed systems. *J. Sound Vib.* (2008) **313**:334-341

[3] M. Zatarain, Z. Dombovari, Stability analysis of milling with irregular pitch tools by the implicit subspace iteration method, Springer-Verlag, Berlin, 2014

[4] M. Zatarain, J. Alvarez, I. Bediaga, J. Munoa, Z. Dombovari, Implicit subspace iteration as an efficient method to compute milling stability lobe diagrams, *Int. J. Adv. Manuf. Technol.* (2015) **77**:597-607

[5] G. M. Hulbert, Time Finite Element Methods for Structural Dynamics, *Int. J. Num. Meth. in Eng.* (1992) **33**:307-331

[6] E. Budak, Y. Altintas, Analytical prediction of chatter stability conditions for multi-degree of systems in milling, *Transactions of ASME Journal of Dynamic Systems Measurement and Control*, (1998) **120**:22-30

[7] P. V. Bayly, J. E. Halley, B. P. Mann, Davies, M. A. Stability of interrupted cutting by temporal finite element analysis. *Journal of Manufacturing Science and Engineering*, (2003) **125**(2):220-225

[8] C. Henninger, P. Eberhard, Improving the computational efficiency and accuracy of the semi-discretization method for periodic delay-differential equations. *European Journal of Mechanics A/Solids* (2008) **27**:975âĂŞ985

[9] T. Insperger, and G. Stepan, Semi-discretization for time-delay systems: stability and engineering applications. *Springer Science Business Media*, 2011. **178**