Theory of harmonic generation in nonlinear elastic waves

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Summary. We present a theory for the dispersion of generated harmonics in a traveling nonlinear wave. The harmonics dispersion relation−derived by the theory−provides direct and exact prediction of the collective harmonics spectrum in the frequency-wavenumber domain, and does so without prior knowledge of the spatial-temporal solution. The new relation is applicable to a family of initial wave functions characterized by an initial amplitude and wavenumber. We demonstrate the theory on nonlinear elastic waves in a homogeneous rod and demonstrate an extension to periodic rods.

Wave motion lies at the heart of many disciplines in the physical sciences and engineering. For example, problems and applications involving light, sound, heat or fluid flow are all likely to involve wave dynamics at some level. In this work, we consider strongly nonlinear wave propagation [1,2] in elastic solids, although the theory presented is in principle applicable to other types of waves such as waves in fluids, gases, and plasma.

We investigate a thick elastic rod admitting longitudinal motion. In the linear limit, this rod is dispersive due to the effect of lateral inertia. The nonlinearity is introduced through either the stress-strain relation and/or the strain-displacement gradient relation. Using a formulation we have developed earlier and demonstrated on thin rods and beams [3], we derive an exact nonlinear dispersion relation for the thick rod. Equation (1) provides the governing nonlinear differential equation considering a linear stress-strain relation and a Green-Lagrange nonlinear relationship between strain and displacement gradient:

$$\partial_{tt}\bar{u} - \partial_{xx}(\alpha\bar{u} + \beta\mathcal{N}(\bar{u}) + \gamma\partial_{tt}\bar{u}), \tag{1}$$

where $\gamma = r^2\nu^2$, r is the polar radius of gyration, and ν is the Poisson's ratio. The Green-Lagrange strain measure is introduced by setting $\alpha = \beta = c^2$ and $\mathcal{N}(\bar{u}) = 3\bar{u}^2/2+\bar{u}^3/2$ where c is the longitudinal speed of sound in the rod. Equation (2) is a statement of the derived nonlinear dispersion relation, and Figure 1 presents it in graphical form with and without accounting for lateral inertia.

$$\omega = c\kappa\sqrt{(2 + 3B\kappa + B^2\kappa^2)/(2 + 2\gamma\kappa^2)}, \tag{2}$$

where ω and κ denote the frequency and wavenumber, respectively, and B represents the wave amplitude.

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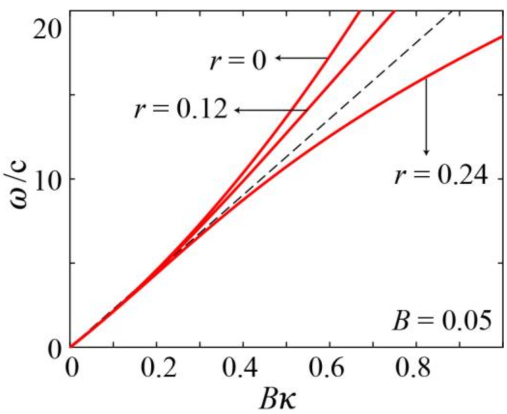

Figure 1. Nonlinear dispersion relation for elastic waves propagating in a thick rod with a radius of gyration r. A non-zero value of r represents the presence of lateral inertial which gives rise to dispersion even when there is no nonlinearity in the system. The wave amplitude is denoted by B. Solid red curves represent dispersion curves for B = 0.05 and dashed black curve represent the linear nondispersive case, i.e., very small value of B.

The derived relation is validated by direct time-domain simulations, examining both instantaneous dispersion (by direct observation) and short-term, pre-breaking dispersion (by Fourier transformations). Figure 2 shows a multi-window overlay of the frequency-wavenumber response obtained by performing a space-time Fourier transform of the simulation field for a collection of hyperbolic secant signals all with an initial amplitude of B = 0.025. Specifically, the contour plot shown is obtained by superimposing the energy spectra of thirty separate simulations for distinct initial wave packets, each following a hyperbolic secant spatial profile and sharing the same amplitude but covering the range of excitation wavenumbers $\kappa_e = 1$ to 30, with increments of 1. What emerges from this exercise is a profile of the fundamental harmonic spanning the various simulations. On the same plot, nonlinear dispersion relation of Eq. (2) is overlaid demonstrating perfect prediction of the simulated nonlinear response [4].

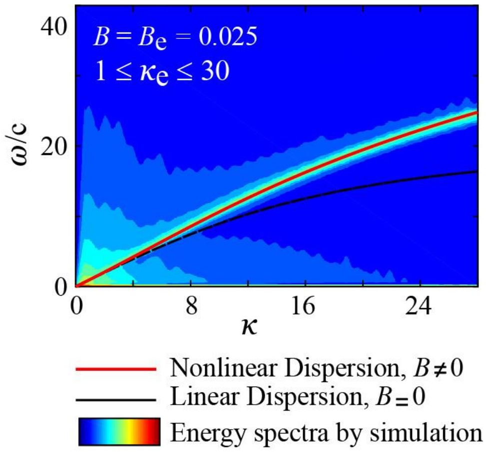

Figure 2. Superposition of harmonics spectra from thirty distinct simulations covering a range of excitation wavenumbers is shown to match perfectly with the general nonlinear dispersion relation of Eq. (2) for the selected value of wave amplitude. Results are for B = 0.025 and r = 0.15.

The study is then extended to a continuous thin rod with a periodic arrangement of material properties [5]. For this problem we introduce a new method that is based on a standard transfer matrix augmented with a nonlinear enrichment at the constitutive material level. This method yields an approximate band structure that accounts for the finite wave amplitude. Finally, we present an analysis on the condition required for the existence of spatial invariance in the wave profile.

References

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