Multiplicity-induced-dominancy for some retarded differential equations

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Summary. The stability analysis of generic retarded differential equations is a difficult question since their asymptotic behavior depends on a non-trivial way on parameters and delays. From a control point of view, the main difficulty is that one has infinitely many spectral roots but only finitely many parameters in the system. A natural question is then to design techniques of assigning a finite number of roots guaranteeing that the rightmost root is among the ones assigned. One such a technique is to place roots of maximal multiplicity, which are often also dominant, a property known as *multiplicity-induced-dominancy* (MID). This paper proves that the MID property holds for some classes of systems by using a priori bounds on the imaginary part of roots on the right half-plane, a suitable factorization of the characteristic equation in integral form, and the analysis of crossing imaginary roots.

Introduction

This paper is interested in linear time-invariant differential equations with delays which can be written under the form

$$y^{(n)}(t)+\sum_{j=0}^{N}\sum_{k=0}^{n-1}\alpha_{j,k}y^{(k)}(t-\tau_j)=0, \tag{1}$$

where n and N are positive integers, $\alpha_{j,k}\in\mathbb{R}$, and $\tau_j\geq 0$. With no loss of generality, we assume $0=\tau_0<\tau_1<\cdots<\tau_N$. Equation (1) is said to be *retarded* since the highest order derivative only appears in the non-delayed term $y^{(n)}(t)$. The stability analysis of time-delay systems has attracted much research effort and is an active field [4, 5, 8, 9, 11]. The asymptotic behavior of (1) can be studied through spectral methods by considering the complex roots of the corresponding characteristic function $\Delta:\mathbb{C}\to\mathbb{C}$ defined for $s\in\mathbb{C}$ by

$$\Delta(s)=s^n+\sum_{j=0}^{N}e^{-s\tau_j}\sum_{k=0}^{n-1}\alpha_{j,k}s^k. \tag{2}$$

Functions such as Δ that can be written under the form $Q(s)=\sum_{j=0}^{\ell}P_j(s)e^{\lambda_j s}$ for some non-zero polynomials with real coefficients $P_0,\dots,P_\ell$ and pairwise distinct real numbers $\lambda_0,\dots,\lambda_\ell$ are called *quasipolynomials*. The integer $D=\ell+\sum_{j=0}^{\ell}d_j$ is called the *degree* of Q, where d_j denotes the degree of P_j. A classical result on quasipolynomials provided in [10, Problem 206.2] implies that the multiplicity of any root of Q does not exceed D.
A root s_0 of Δ is said to be *(strictly) dominant* if every other root of Δ has a real part (strictly) smaller than $\operatorname{Re} s_0$. It has been observed in several works that real roots of maximal multiplicity tend to be dominant, a property known as *multiplicity-induced-dominancy* (MID) (see, e.g., [1, 3] for the case $N=1$ and $n=2$ using approaches based on factorization and Cauchy's argument principle). The interest in considering multiple roots does not rely on the multiplicity itself, but on its connection with dominance and its implications for stability analysis and control design.

Main results

The main results of this paper prove that the MID property holds for some classes of systems under the form (1).

Theorem 1 *Consider the quasipolynomial Δ given by* (2) *in the case $N=1$. Let $s_0\in\mathbb{R}$. The number s_0 is a root of multiplicity $2n$ of Δ if and only if, for every $k\in\{0,\dots,n-1\}$,*

$$\begin{cases}\alpha_{0,k}=\binom{n}{k}(-s_0)^{n-k}+(-1)^{n-k}n!\displaystyle\sum_{j=k}^{n-1}\binom{j}{k}\binom{2n-j-1}{n-1}\frac{s_0^{j-k}}{j!\tau_1^{n-j}},\\ \alpha_{1,k}=(-1)^{n-1}e^{s_0\tau_1}\displaystyle\sum_{j=k}^{n-1}\frac{(-1)^{j-k}(2n-j-1)!}{k!(j-k)!(n-j-1)!}\frac{s_0^{j-k}}{\tau_1^{n-j}}.\end{cases} \tag{3}$$

Moreover, if (3) *is satisfied, then s_0 is a strictly dominant root of Δ.*

Thanks to the change of variables $z=\tau_1(s-s_0)$ transforming $\Delta(s)$ into $\hat{\Delta}(z)=\tau^n\Delta(s_0+\frac{z}{\tau_1})$, it suffices to prove Theorem 1 in the case $s_0=0$ and $\tau_1=1$. Its first part can be obtained by solving the system of linear equations $\hat{\Delta}^{(\ell)}(0)=0$ for $\ell\in\{0,\dots,2n-1\}$. As for the second part, when (3) is satisfied and $z=0$ is a root of multiplicity $2n$ of $\hat{\Delta}$, one can factorize $\hat{\Delta}$ as $\hat{\Delta}(z)=z^{2n}I(z)$, where $I(z)$ is an entire function which can be written as an integral, for instance $I(z) = \int_0^1 t(1-t)^2 e^{-zt}\,\mathrm{d}t$ when $n = 2$. A direct study of I, making use of the a priori information that roots z of $\hat{\Delta}$ in the closed right half-plane $\mathbb{C}_+$ must satisfy $|\operatorname{Im} z| \leq 2\pi$, shows that it cannot have roots in $\mathbb{C}_+$. A detailed proof is provided in [6].

Theorem 2 *Consider the quasipolynomial* Δ *given by* (2) *in the case* $N = 1$ *and* $n = 2$. *Let* $s_0 \in \mathbb{C}$, $\sigma_0 = \operatorname{Re} s_0$, *and* $\theta_0 = \operatorname{Im} s_0$ *and assume that* $\theta_0 \neq 0$. *The numbers* s_0 *and* $\overline{s}_0$ *are roots of multiplicity* 2 *of* Δ *if and only if*

$$\begin{cases} \alpha_{0,0} = \sigma_0^2 + 2\sigma_0\theta_0 \dfrac{\tau_1\theta_0 - \sin(\tau_1\theta_0)\cos(\tau_1\theta_0)}{\tau_1^2\theta_0^2 - \sin^2(\tau_1\theta_0)} + \theta_0^2 \dfrac{\tau_1^2\theta_0^2 + \sin^2(\tau_1\theta_0)}{\tau_1^2\theta_0^2 - \sin^2(\tau_1\theta_0)}, \\ \alpha_{0,1} = -2\sigma_0 - 2\theta_0 \dfrac{\tau_1\theta_0 - \sin(\tau_1\theta_0)\cos(\tau_1\theta_0)}{\tau_1^2\theta_0^2 - \sin^2(\tau_1\theta_0)}, \\ \alpha_{1,0} = 2\theta_0 e^{\sigma_0\tau_1}\left(\sigma_0 \dfrac{\sin(\tau_1\theta_0) - \tau_1\theta_0\cos(\tau_1\theta_0)}{\tau_1^2\theta_0^2 - \sin^2(\tau_1\theta_0)} - \dfrac{\tau_1\theta_0^2\sin(\tau_1\theta_0)}{\tau_1^2\theta_0^2 - \sin^2(\tau_1\theta_0)}\right), \\ \alpha_{1,1} = 2\theta_0 e^{\sigma_0\tau_1} \dfrac{\tau_1\theta_0\cos(\tau_1\theta_0) - \sin(\tau_1\theta_0)}{\tau_1^2\theta_0^2 - \sin^2(\tau_1\theta_0)}. \end{cases} \tag{4}$$

Moreover, if (4) *is satisfied, then* s_0 *and* $\overline{s}_0$ *are a pair of strictly dominant roots of* Δ.

Similarly to Theorem 1, one may reduce the proof of Theorem 2 to the case $\sigma_0 = 0$ and $\tau_1 = 1$, with (4) being again proved by solving a linear system of equations. As $\theta_0 \to 0$, (4) converges (3) with $n = 2$, and one can thus prove dominance of the complex conjugate pair $\pm i\theta_0$ by proving that, as θ_0 increases from 0, no roots of $\hat{\Delta}$ may lie on the imaginary axis other than $\pm i\theta_0$, and hence no roots can cross to the right half-plane. See [7] for a detailed proof.

Theorem 3 *Consider the quasipolynomial* Δ *given by* (2) *in the case* $N = 2$ *and* $n = 1$. *Let* $s_0 \in \mathbb{R}$. *The number* s_0 *is a root of multiplicity* $2n$ *of* Δ *if and only if*

$$\alpha_{0,0} = -s_0 - \frac{1}{\tau_1} - \frac{1}{\tau_2}, \qquad \alpha_{10} = e^{s_0\tau_1}\frac{\tau_2}{\tau_1(\tau_2 - \tau_1)}, \qquad \alpha_{20} = -e^{s_0\tau_2}\frac{\tau_1}{\tau_2(\tau_2 - \tau_1)}. \tag{5}$$

Moreover, if (5) *is satisfied, then* s_0 *is a strictly dominant root of* Δ.

One obtains (5) using the same arguments from Theorems 1 and 2. The second part of the statement can be proved by considering the limit $\tau_2 \to \tau_1$ and proving that s_0 is dominant for the limiting quasipolynomial and that, as τ_2 increases, no roots other than s_0 may have real part s_0, excluding the possibility of any other root becoming dominant.

Conclusion

We further explored the MID property for generic single-delay retarded systems of arbitrary order, showing that a real spectral value with maximal multiplicity is necessarily dominant. For a scalar equation with two delays, it is shown that the MID property still applies. Further, in the second-order case, we contributed by extending the MID property for complex conjugate pairs of spectral values.
In recent studies, the applicability of the MID property in reduced-complexity delayed controller design was shown, where the attenuation of the dominant vibrating modes of flexible mechanical structures was considered [2]. In future work, the MID property will be further exploited in the problem of vibration quenching.

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