Non-stationary dynamics of the sine-lattice consisting of three pendulums (trimer)

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Summary. We present an analysis of both low- and high-amplitude oscillations of nonlinearly coupled trimer, even when the quasi-linear approximation cannot be applied. The described models are fundamental for many areas of Mechanics and Physics (paraffin crystals, DNA molecules etc.). We obtained the conditions of stability of basic stationary solutions corresponding to nonlinear normal modes (NNMs). Supposing the NNMs resonant interaction we introduce a “slow” time-scale which determines a characteristic time of the energy exchange between the pendulums. Introducing the angle variables, we reduce the considered phase space.

Introduction

Dynamics of coupled nonlinear oscillators attracts the growing interest of scientific community because of its fundamental meaning and various applications.
We will focus on the large-amplitude oscillations but not rotations of the pendulums. In current paper, in contrast to many works devoted to interacting nonlinear oscillators, non-linearity of both the pendulums and the coupling between them are not assumed to be small. Thus, research methods that involve quasi-linearity and the presence of a small parameter characterizing nonlinearity and/or coupling are not applicable. To overcome this difficulty a semi-inverse method was proposed [1]. Using this method and LPT concept the system of two identical linearly and strongly nonlinearly coupled pendulums was examined under different oscillation amplitudes [2]. Stationary and non-stationary transitions leading to a qualitative change in the dynamic behavior of the system were analytically described. This work continues the previous investigations for the more complex case, when the coupling between the pendulums cannot be assumed, and the number of degrees of freedom is equal three.

The model and asymptotic procedure

Hamiltonian of the system of three identic pendula coupled via cosine potential in the dimensionless form can be represented in the following form:

$$H=\sum_{j=1,2,3}\left(\frac{1}{2}\left(\frac{dq_j}{dt}\right)^2+\left(1-\cos(q_j)\right)\right)+\beta\left(1-\cos(q_2-q_3)\right)+\beta\left(1-\cos(q_2-q_1)\right) \tag{1}$$

We suppose that NNMs frequencies are close to each other, and the motion of the system happens with the frequency which is also close to those of Nonlinear Normal Modes (NNMs) of the system. The assumption of the closeness of the motion to the resonance with single frequency allows application of the semi-inverse approach, which was already used in our earlier works. Closeness to resonance allows us to introduce a small parameter and further apply a multi-scale procedure. In the slow time-scale we obtain a system of equations with new form of non-linearity:

$$\begin{aligned}
&i\frac{d}{d\tau_1}\varphi_1+\mu\left(-\frac{\omega}{2}\varphi_1+\frac{1}{\sqrt{2\omega}}J_1\left(\sqrt{\frac{2}{\omega}}|\varphi_1|\right)\frac{\varphi_1}{|\varphi_1|}\right)+\frac{\beta_0}{\sqrt{2\omega}}J_1\left(\sqrt{\frac{2}{\omega}}|\varphi_1-\varphi_2|\right)\frac{\varphi_1-\varphi_2}{|\varphi_1-\varphi_2|}=0,\\
&i\frac{d}{d\tau_1}\varphi_2+\mu\left(-\frac{\omega}{2}\varphi_2+\frac{1}{\sqrt{2\omega}}J_1\left(\sqrt{\frac{2}{\omega}}|\varphi_2|\right)\frac{\varphi_2}{|\varphi_2|}\right)+\frac{\beta_0}{\sqrt{2\omega}}J_1\left(\sqrt{\frac{2}{\omega}}|\varphi_2-\varphi_1|\right)\frac{\varphi_2-\varphi_1}{|\varphi_j-\varphi_{3-j}|}+\frac{\beta_0}{\sqrt{2\omega}}J_1\left(\sqrt{\frac{2}{\omega}}|\varphi_2-\varphi_3|\right)\frac{\varphi_2-\varphi_3}{|\varphi_2-\varphi_3|}=0,\\
&i\frac{d}{d\tau_1}\varphi_3+\mu\left(-\frac{\omega}{2}\varphi_3+\frac{1}{\sqrt{2\omega}}J_1\left(\sqrt{\frac{2}{\omega}}|\varphi_3|\right)\frac{\varphi_3}{|\varphi_3|}\right)+\frac{\beta_0}{\sqrt{2\omega}}J_1\left(\sqrt{\frac{2}{\omega}}|\varphi_3-\varphi_2|\right)\frac{\varphi_3-\varphi_2}{|\varphi_3-\varphi_2|}=0,
\end{aligned} \tag{2}$$

where J_1 is Bessel function of the first kind. Using such reptresentation we can obtain analytical description of the NNMs' frequencies in the system. We see a very good agreement for the initial excitation up to Q=9/10 π . As we expected the frequency of the oscillations decreases with the increase of the initial excitation (due to the ‘soft’ nonlinearity effect). We should also emphasize that our assumption on the closeness of the NNMs frequencies appears to be valid for a wide range of the parameters and initial conditions.

Non-stationary dynamics: Poincaré maps study

To proceed with the study of the phase space we intend to reduce the dimensionality of the model. Similarly to the system of two pendula [2] asymptotic system (2) possesses an additional integral of motion

$$X=\sum_{k=1}^{3}|\varphi_k|^2$$

It allows us to introduce spherical coordinates

$$\begin{cases} \varphi_1 = \sqrt{X}\cos\theta\cos\varphi\, e^{i\delta_1} \\ \varphi_2 = \sqrt{X}\sin\theta\, e^{i\delta_2} \\ \varphi_3 = \sqrt{X}\cos\theta\sin\varphi\, e^{i\delta_3} \end{cases} \tag{3}$$

and using the fact that only the relative phases have physical meaning we reduce the system's dimentionality: $\Delta_{12} = \delta_1 - \delta_2,\ \ \Delta_{23} = \delta_2 - \delta_3$. The systems Hamiltonian:

$$H_{4D} = \mu\left(\frac{1}{4}Q^2\omega^2 - J_0\left(Q\sin\theta\right) - J_0\left(Q\cos\theta\cos\varphi\right) - J_0\left(Q\cos\theta\sin\varphi\right)\right) - \beta_0 J_0\left(Q\sqrt{S_1}\right) - \beta_0 J_0\left(Q\sqrt{S_2}\right). \tag{4}$$

System (8) is four-dimensional, and its dimensionality can be reduced using Hamiltonian (9), but the system remains non-integrable even in the slow time scale. However, the Poincaré sections analysis of the reduced system can be provided. The section plane was defined as $\theta = 1.53$, Δ_{23} was defined from (4) as $H = h(\beta_0)$.

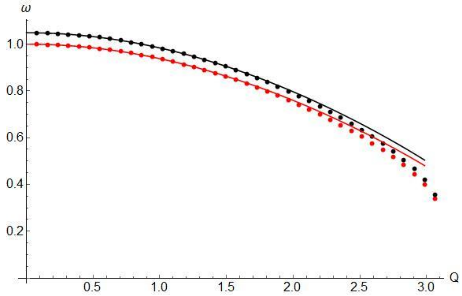

Figure 1: NNMs of the asymptotic system (solid lines) and comparison with the numerical results of the full system (dotted lines)

We have constructed Poincaré maps for different values of the coupling parameter *β*. For low values of coupling the dynamics is regular; there are two stationary points on the map C_1 and C_2, which correspond to quasiperiodic dynamical regimes with energy localization on one of the side-oscillators (see Fig 2a). If the coupling increases the chaotic regimes occupy the most of the phase space, then a new periodic regime C_3 of regular energy transport from one side-element to another appears in the chaotic region (see Fig 2b). We remind that this regime is called LPT, which is a phase trajectory that shows the possibility of the intensive regular energy exchange between the two edges of the short chain.

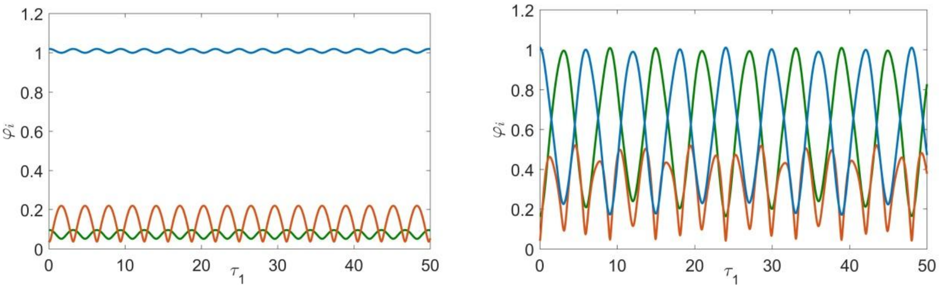

Figure 2: Time-evolutions of different regimes of system (4): θ=1.53, ε=0.1, μ=10, a) β0 = 0.5, initial conditions correspond to C1; b) β0 = 2, initial conditions correspond to C3

Conclusions

We report the study of the non-stationary dynamics in the system of three pendulums coupled by cosine potential. In the earlier works the intensive beatings were reported in the systems with more than two degrees of freedom with the periodic boundary conditions but they characterize the energy exchange not between the coherent domains. Present work extends the phenomenon of the intensive periodic energy exchange between the two ends of the short oscillatory chains with more than two degrees of freedom. The physical meaning of the regime with excitation mostly localized on the side-element is similar to that of the discrete breather in the long chain of nonlinear oscillators

Acknowledgements

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References

[1] Manevitch LI., Smirnov VV (2016) Semi-inverse Method in Nonlinear Dynamics, in Proceedings of 5th International Conference "Nonlinear Dynamics", Kharkov, 28-37.

[2] Kovaleva M.A., Smirnov V.V., Manevitch LI. Stationary and nonstationary dynamics of the system of two harmonically coupled pendulums Rus. J. Nonlin. Dyn., 2017, vol. 13, no. 1, pp. 105–115 (Russian)

[3] Kovaleva M, Smirnov V., Manevitch L. I. Nonstationary dynamics of the sine lattice consisting of three pendula (trimer) // Physical Review E - Statistical, Nonlinear, and Soft Matter Physics. 2019. Vol. 99. No. 012209. P. 012209-1-012209-12.