Coexistence of conservative and dissipative dynamics in forced vibro-impact oscillator with Amonton-Coulomb friction

Pavel Kravetc*, Oleg Gendelman* and Dmitrii Rachinskii †
**Faculty of Mechanical Engineering, Technion – Israel Institute of Technology, Haifa, 3200003, Israel*
†*Department of Mathematical Sciences, The University of Texas at Dallas, Richardson, Texas, 75080, USA*

Summary. The qualitative differences in the behaviors of Hamiltonian and dissipative systems are profound. In particular, in Hamiltonian systems the existence of asymptotically stable solutions is impossible due to the Liouville's theorem. By contrast, dissipative systems usually exhibit eventual attraction of all phase trajectories to ω-limit sets such as fixed points, (quasi-) periodic or strange attractors. In this work we present a class of dissipative mechanical systems which demonstrate mixed global dynamics, i.e., coexistence of both attracting and Hamiltonian-like behaviors. Our main model is a well-known system of harmonically forced vibro-impact oscillator with Amonton-Coulomb friction. In a vast majority of previous studies, it also includes viscous friction, and the global dynamics of its state space is governed by the aforementioned attractors. However, if we omit viscous friction, we observe that the state space is divided into regions with "regular" attraction, as well as regions with profoundly Hamiltonian dynamics, e.g., KAM-like tori. We show that such local Hamiltonian behavior occurs for the phase trajectories with non-vanishing velocities. The stability analysis of the periodic solutions confirms the above statement. We also demonstrate that similar mixed global dynamics can be observed in a broader class of models including systems with x-dependent potentials as well as multi-particle systems.

Introduction

Model formulation

We consider a single-degree-of-freedom unit mass particle which displacement is constrained by two rigid walls ($l < x(t) < r$). The particle is subject to external sinusoidal forcing of period $T = 2\pi/\omega$ and the Amonton-Coulomb dry friction [1, 2]. The equation for motions $l < x(t) < r$ is

$$\ddot{x} + f \operatorname{sgn}(\dot{x}) = F \cos(\omega t). \tag{1}$$

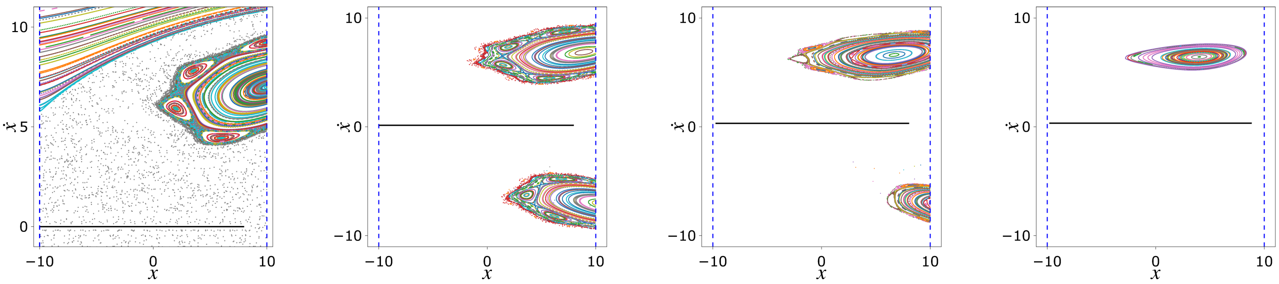
We assume that $F > f$, since otherwise any motion comes to stop. When $x(t) = r, l$ the following impact rule [3] is applied:

$$\dot{x}(t^-) = -\dot{x}(t^+). \tag{2}$$

Dynamics of system (1)–(2), and its dependence on parameters, will be illustrated by the global phase portrait of the stroboscopic time T map $\Phi : (x(0),\ \dot{x}(0)) \mapsto (x(T),\ \dot{x}(T))$. In particular, a fixed point of Φ corresponds to a periodic solution of system (1)–(2).

Numerical investigations

A typical phase portrait of Φ with $f = 0$ is shown on Figure 1a. It presents an invariant island containing a fixed point and periodic orbits which are surrounded by invariant curves representing quasiperiodic motions. Another region of quasiperiodic motions is observed for large velocities. Between these two regions there is a chaotic sea (gray dots). Furthermore, a horizontal segment of fixed points represents periodic solutions without impacts $x(t) = -\frac{F}{\omega 2} \cos(\omega t) + C$.



(a) $f = 0$ (b) $f = 0.1$ (c) $f = 0.3$ (d) $f = 0.5$

Figure 1: Evolution of the phase portrait with increasing friction. Other parameters are the following: $F = 1, \omega = 1, R := r - l = 20$.

Introducing friction drastically changes the phase portrait (Figures 1b–d). In particular, the segment of fixed points corresponding to periodic solutions without impacts becomes an attractor. However, the invariant island of Hamiltonian dynamics persists, thus, we observe coexistence of dissipative and conservative behaviors. The trajectories from the invariant island have a non-vanishing velocity, and, therefore, we call such solutions *non-sticking*.

Analysis

Hamiltonian dynamics of system (1)–(2) can be explained by lifting it to the unconstrained Hamiltonian system

$$\ddot{q} + \frac{\partial V}{\partial q} = \frac{f}{R}, \qquad V(t,\ q) = \frac{F}{R}\cos(\omega t)W(q), \qquad W(q) = \begin{cases} q, & 0 \le q < 1, \\ 2-q, & 1 \le q < 2, \end{cases} \qquad W(q+2) = W(q). \tag{3}$$

Any non-sticking solution of (3) is mapped to a non-sticking solution of (1)–(2) by the simple relationship

$$x = R \cdot W(q) + l. \tag{4}$$

Therefore, if all solutions starting from a domain Ω of the state space of system (1)–(2) are non-sticking and Ω is invariant for the time T map Φ of this system, then Ω is a region of Hamiltonian dynamics. In particular, at least a small invariant region Ω of Hamiltonian dynamics exists around every fixed point of non-degenerate non-resonant center type.

Stability analysis

Stability and type of a fixed point and the corresponding periodic solution is determined using the Floquet theory as follows. Any trajectory of (1)–(2) can be represented as a sequence of motions and events of the following types: *free flight* ($\dot{x} \neq 0$), *sticking* ($\dot{x} = 0$ over a nonzero interval of time), *reflection from the wall* ($x = r, l$) and *a turning point* (change of sign of $\dot{x}$). Thus, the Jacobi matrix Φ' can be obtained as a product of corresponding matrices (matrix related to an instantaneous event is known as *saltation matrix* [4]). In particular, it is easy to show that matrices for free flights and reflections of the walls have unit determinants. Thus, non-sticking trajectories, $\det(\Phi') = 1$ which corresponds to the phase area preservation.

Extensions

Mixed global dynamics can also be observed in systems with x-dependent potentials as well as systems with higher degrees of freedom. For instance, one can consider system of $n \geq 2$ elastically colliding particles with masses m_i placed between the same two walls. Again, in order to spot regions of Hamiltonian-like behavior we should check non-sticking periodic trajectories of non-resonant center type. The linear stability analysis can be easily extended to the case of many particles, however, in order to be volume-preserving the Jacobian Φ' should be symplectic. It turns out that the saltation matrix S related to the particle collision is not necessary symplectic. However, if the product of velocities of the two colliding particles has the same sign before and after the collision, then the corresponding saltation matrix (for $n = 2$)

$$S = \begin{bmatrix} A^T & 0 \\ B & A \end{bmatrix} \quad \text{with} \quad A = \frac{1}{m_1 + m_2}\begin{bmatrix} m_1 - m_2 & 2m_1 \\ 2m_2 & m_2 - m_1 \end{bmatrix} \quad \text{and} \quad B = \frac{2F\cos(\omega t_*)(m_1 - m_2)m_1 m_2}{m_1 + m_2\left(m_2\dot{x}_1(t_*^-) - m_1\dot{x}_2(t_*^-)\right)}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}, \tag{5}$$

is symplectic. When $m_1 = 1$, $m_2 = 0.9$, $F = 1$, $\omega = 2\pi$, $f = 0.005$, $R = 2$, the two-particle system has a $3T$-periodic solution which linearization is symplectic. Figure 2 presents invariant tori in a neighborhood of this solution. The tori coexist with an attractor consisting of periodic solutions without collisions or impacts.

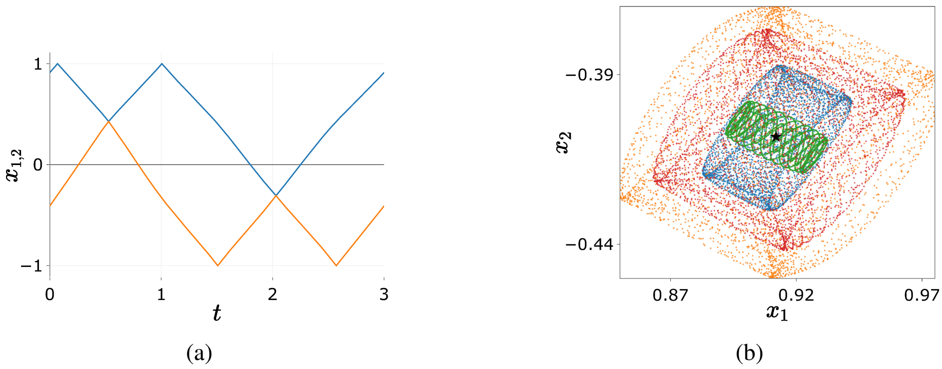


Figure 2: Panel (a): Time trace of a $3T$-periodic solution of the two-particle system with $m_1 = 1$ (blue), $m_2 = 0.9$ (orange). Panel (b): Projection of the phase portrait of the time T map Φ onto the (x_1, x_2)-plane exhibiting several invariant tori near a period 3 point (star).

References

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