On the Nonreciprocal Dynamics of Bilinearly Coupled Oscillators

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Summary. This work explores nonreciprocity in the dynamic response of oscillators that are coupled by a bilinear elastic force and subject to external harmonic drive. The conditions for breaking reciprocity and the influence of different parameters on the steady-state nonreciprocal dynamics of the system are described. The manifestation of nonreciprocity in the time profile of the response is discussed in terms of the amplitudes of oscillation, phase differences and time-independent shifts. While the main focus is on the frequency-preserving response regime, the nonreciprocal response beyond the weakly nonlinear regime is also addressed.

Introduction

There has been an incredible interest in the recent years to explore scenarios in which reciprocity theorems do not hold [1]. A main motivation is that deviation from reciprocity can lead to asymmetric propagation of elastic waves for opposite directions. Reciprocity theorems generally apply to linear, time-invariant systems [2]. The breakdown of reciprocity may take place, for example, in systems with time-dependent (active) [3] or nonlinear (passive) [4] material properties.
The focus of this work is on nonreciprocity caused by nonlinearity. In particular, the nonreciprocal dynamics that results from bilinear elasticity is studied. A bilinear spring exhibits two different values of stiffness with a transition occurring at a critical strain. Bilinear stiffness is used, among other problems, for modeling systems involving contact and friction. A system with bilinear elasticity, due to its non-smooth character, can exhibit complex nonlinear dynamic behavior [5].
This work studies the conditions that lead to the emergence of nonreciprocal dynamics in bilinearly coupled oscillators subject to external harmonic excitation.

Bilinearly Coupled Oscillators

The following normalized equations govern the dynamics of the system considered here

$$\begin{cases} \ddot{x}_1 + c\dot{x}_1 + x_1 + k_c(x_1 - x_2) = F_1\cos(\Omega t) \\ (1+\mu)\ddot{x}_2 + c\dot{x}_2 + x_2 + k_c(x_2 - x_1) = F_2\cos(\Omega t) \end{cases}, \quad k_c = \begin{cases} 1+b\,, & x_1 - x_2 > d \\ 1\,, & x_1 - x_2 < d \end{cases} \tag{1}$$

where d is the offset above which bilinearity is triggered. The steady state dynamic response is computed as the periodic orbits of Eq. (1) using the numerical continuation software package AUTO [6]. The bilinear stiffness is regularized using a hyperbolic tangent function to maintain smoothness. A very large coefficient is used for regularization to ensure the results are representative of the bilinear system.
To investigate reciprocity, two configurations need to be considered: (i) *forward*, where $F_1 = P$, $F_2 = 0$ and the steady response of x_2 is monitored; (ii) *backward*, where $F_1 = 0$, $F_2 = P$ and the steady response of x_1 is monitored. A reciprocal response is characterized by $x_2^f = x_1^b$, where the superscripts f and b denote the response in the forward and backward configurations, respectively. To better analyze the results, the steady time-periodic response of each oscillator is decomposed into a constant (time-independent) and oscillating component: $x_i(t) = C_i + y_i(t)$, where $C_i = (1/T)\int_0^T x_i(t)dt$ with T denoting the period of oscillations. Furthermore, the amplitude of oscillations is defined as A_i, where $A_i^2 = (2/T)\int_0^T y_i^2(t)dt$. A difference norm M is used as a quantitative measure of nonreciprocity: $M = (1/T)\int_0^T (x_2^f - x_1^b)^2 dt$.

Nonreciprocal Dynamics

Fig. 1 shows the steady response of the system for $\mu = 1$ and $b = 1$. A damping coefficient of $c = 0.008$ is chosen. A non-zero value of $d = 0.1$ is chosen for the offset; accordingly, the response of the system is linear for very small values of the driving amplitude. The driving amplitude $P = 0.0015$ is just high enough to trigger the transition into bilinearity (i.e., $x_1 - x_2 > d$) for one of the configurations. Panel (a) shows the normalized amplitude of oscillations $\log_{10}(A_i/P)$ as a function of Ω for the forward and backward configurations. As expected, the first appearance of nonlinear behavior occurs near the out-of-phase mode because $(x_1 - x_2)$ has a larger value there. The inspection of A_i near this resonance in panel (b) shows that while the system is behaving linearly in the backward configuration, the response of the forward configuration has clearly entered the nonlinear regime. The initial effect of bilinearity is to increase the frequency of the second mode (because $b > 0$). Panel (c) shows the difference norm M for the same frequency range as in panel (b). The time profile of three points are shown in panels (d)-(f), with the corresponding points marked by a triangle, square and circle in panels (b) and (c). Panel (d) corresponds to the peak oscillation amplitude of the backward configuration. Panel (e) corresponds to equal oscillation amplitudes for the forward and backward configuration. Note, however, that it corresponds to a large phase difference between the two configurations, which is responsible for the large value of M; see [7] for a detailed discussion. Panel (f) corresponds to the peak oscillation amplitude of the forward configuration. The backward configuration is off-resonance at this point, explaining the large difference between the amplitudes.

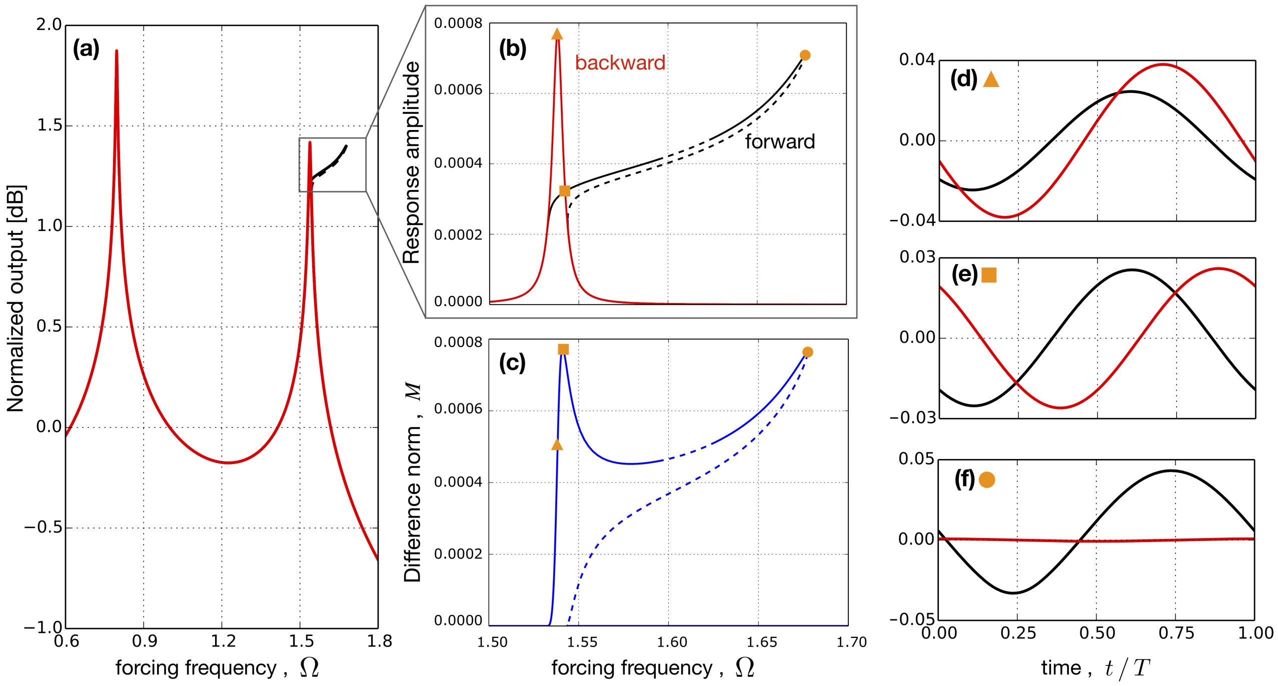

Figure 1: The steady-state dynamic response of the system for $\mu = 1$ and $b = 1$.

The bilinear nature of the coupling force causes nonreciprocity in the system, manifested by a large difference in the amplitude of oscillations (panel (f)), as well as a significant phase difference between the backward and forward configurations (panel (e)). The bilinear response may also exhibit a time-independent component (a DC shift). In particular, the shift in the response is different for the forward and backward configurations, contributing to the nonreciprocity of the response. This shift can be seen in panel (f) for the forward configuration. In the weakly nonlinear regime, the DC shift in the response is less significant than the shift in the resonance frequencies (i.e., it appears in a higher order of correction). It is important to note the significant influence of b and μ on nonreciprocity. b controls the degree of nonlinearity and its type (softening or hardening). In the limit of very large b, the coupling force is effectively modeling an impact problem [5]. Having a non-zero value for μ is essential for violating reciprocity because it is the parameter than breaks the symmetry; nonlinearity and asymmetry are both needed for breaking reciprocity in a passive dynamical system. The values used here for b and μ (as well as for c and d) can be realized in experiments.

At larger driving amplitudes (equivalently, at higher values of b or lower values of d), the backward configuration also exhibits nonlinear behavior in the out-of-phase mode. Beyond the weakly nonlinear response, we also observe some of the typical bifurcations in a bilinear system such as period doubling and subharmonic resonances. A period-doubling bifurcation is already detectable in Fig. 1(b), though it is not further explored here. As expected, these nonlinear behaviors are more readily accessible through the out-of-phase mode of the system.

Conclusions

Bilinearly coupled oscillators can exhibit steady nonreciprocal dynamics when subject to external harmonic drive. The nonreciprocal regime can be reached at relatively small values of driving amplitude, degree of bilinearity and asymmetry, making the mechanical system a good candidate for experiments. The nonreciprocal behavior manifests in the time domain as different amplitudes of oscillation, phases and time-independent shifts. Typical instabilities such as period-doubling bifurcations may be observed and further exploited for nonreciprocity. This work highlights the potential of bilinear elasticity in realizing passive nonreciprocity in mechanical systems.

References

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